



# On global regular solutions to magnetohydrodynamics in axi-symmetric domains

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**Abstract.** We consider mhd equations in three-dimensional axially symmetric domains under the Navier boundary conditions for both velocity and magnetic fields. We prove the existence of global, regular axi-symmetric solutions and examine their stability in the class of general solutions to the mhd system. As a consequence, we show the existence of global, regular solutions to the mhd system which are close in suitable norms to axi-symmetric solutions.

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## 1. Introduction

We examine viscous, incompressible magnetohydrodynamics (mhd) flows in axially symmetric domains in  $\mathbb{R}^3$ . The governing equations read

$$\begin{aligned} \mathbf{v}_{,t} + (\mathbf{v} \cdot \nabla) \mathbf{v} - \nu \Delta \mathbf{v} + \nabla q - (\mathbf{H} \cdot \nabla) \mathbf{H} &= \mathbf{f} & \text{in } \Omega^{kT} := \Omega \times (kT, (k+1)T) \\ \operatorname{div} \mathbf{v} &= 0 & \text{in } \Omega^{kT}, \\ \mathbf{H}_{,t} + (\mathbf{v} \cdot \nabla) \mathbf{H} - (\mathbf{H} \cdot \nabla) \mathbf{v} - \mu \Delta \mathbf{H} &= \mathbf{0} & \text{in } \Omega^{kT}, \\ \operatorname{div} \mathbf{H} &= 0 & \text{in } \Omega^{kT}, \end{aligned} \quad (1.1)$$

where  $k$  is a natural number including 0 and:

- $\mathbf{v} = (v_1(x, t), v_2(x, t), v_3(x, t)) \in \mathbb{R}^3$  is the velocity of the fluid,
- $q = p(x, t) + \frac{|\mathbf{H}|^2}{2} \in \mathbb{R}$  is the total pressure,
- $p = p(x, t) \in \mathbb{R}$  is the pressure,
- $\mathbf{H} = (H_1(x, t), H_2(x, t), H_3(x, t)) \in \mathbb{R}^3$  is the magnetic field,
- $\mathbf{f} = (f_1(x, t), f_2(x, t), f_3(x, t)) \in \mathbb{R}^3$  is the external force field,
- $x = (x_1, x_2, x_3)$  is the Cartesian system of coordinates,
- $\nu > 0$  and  $\mu > 0$  are constant viscosity and resistivity coefficients, respectively.

Let  $S := \partial\Omega$  denote the boundary of  $\Omega$ . Then, we supplement (1.1) with the following boundary conditions

$$\begin{aligned} \mathbf{v} \cdot \mathbf{n} &= 0, \\ \mathbf{n} \times \operatorname{rot} \mathbf{v} &= \mathbf{0} \end{aligned} \quad (1.2)$$

and

$$\begin{aligned} \mathbf{H} \cdot \mathbf{n} &= 0, \\ \mathbf{n} \times \operatorname{rot} \mathbf{H} &= \mathbf{0} \end{aligned} \quad (1.3)$$

on  $S^{kT} = S \times (kT, (k+1)T)$ ,  $k \in \mathbb{N} \cup \{0\}$ , where  $\mathbf{n}$  is the unit outward vector normal to  $S$ . As the initial conditions, we take

$$\begin{aligned}\mathbf{v}|_{t=kT} &= \mathbf{v}(kT) \quad \text{in } \Omega, \\ \mathbf{H}|_{t=kT} &= \mathbf{H}(kT) \quad \text{in } \Omega.\end{aligned}\tag{1.4}$$

Throughout this paper, we assume that  $\Omega$  is an axially symmetric, bounded domain located in a positive distance from the axis of symmetry. Its geometry can be easily expressed in cylindrical coordinates  $(r, \varphi, z)$  which are introduced in the standard way through a mapping  $\Phi$ ,  $x = (x_1, x_2, x_3) \stackrel{\Phi}{=} (r \cos \varphi, r \sin \varphi, z)$ . Then, the basis vectors read

$$\begin{aligned}\mathbf{e}_r &= \partial_r \Phi = (\cos \varphi, \sin \varphi, 0), \\ \mathbf{e}_\varphi &= \partial_\varphi \Phi = (-\sin \varphi, \cos \varphi, 0), \\ \mathbf{e}_z &= \partial_z \Phi = (0, 0, 1).\end{aligned}$$

Let  $x_3$  be the line of intersection of two planes:  $P(\varphi)$  and  $x_2 = 0$ , where  $\varphi$  is the dihedral angle between  $P(\varphi)$  and  $x_2 = 0$ . Let  $\psi(r, z) = 0$  be a closed curve in the plane  $P(\varphi)$  such that  $0 < a \leq r \leq b$  (e.g. one could take  $\psi(r, z) = (r - a_0)^2 + z^2 - r_0^2 = 0$ , where  $a_0 > r_0$ , so  $r \in [a_0 - r_0, a_0 + r_0]$  and  $|z| \leq r_0$ ). Then, we define  $\Omega$  as a solid of revolution around the  $x_3$ -axis

$$\Omega := \{(r, \varphi, z) : \psi(r, z) < 0, \quad \varphi \in [0, 2\pi]\}.\tag{1.5}$$

Let  $\varphi_0 \in [0, 2\pi]$  be fixed. We introduce

$$\Omega_{\varphi_0} = P(\varphi_0) \cap \Omega, \quad S_{\varphi_0} = \partial \Omega_{\varphi_0}.\tag{1.6}$$

Then

$$\mathbf{n}|_{S_{\varphi_0}} = \frac{\nabla \psi}{|\nabla \psi|} = \frac{1}{|\nabla \psi|} \left[ \frac{\psi_{,r}}{r} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} + \psi_{,z} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]\tag{1.7}$$

where  $|\nabla \psi| = \sqrt{\psi_{,r}^2 + \psi_{,z}^2}$ . Let us note that the right-hand side in the above expression does not depend on  $\varphi_0$ , which allows us to utilize the Cartesian coordinate system.

Our main goal (cf. Theorem 3) is to prove the existence of global, regular solutions to problem (1.1)–(1.4) without any assumptions on smallness of the initial and the external data, however, fulfilling certain geometrical constraints which we will describe in the subsequent paragraphs. Our proof is based on stability reasoning: we construct a special solution and examine its stability in the class of solutions to (1.1). As a by-product, we obtain a solution (1.1). This method has been recently utilized in e.g. [1, 2].

The first step of our work is a construction of a special, axially symmetric solution  $(\mathbf{v}_s, \mathbf{H}_s)$ . By *axially symmetric*, we mean

$$\partial_\varphi v_{sr} = \partial_\varphi v_{s\varphi} = \partial_\varphi v_{sz} = \partial_\varphi H_{sr} = \partial_\varphi H_{s\varphi} = \partial_\varphi H_{sz} = 0,\tag{1.8}$$

thus  $\mathbf{v}_s = \mathbf{v}_s(r, z, t)$ ,  $\mathbf{H}_s = \mathbf{H}_s(r, z, t)$ . Since  $\Omega$  is located in a positive distance from its axis of symmetry the construction of this special solution is much easier because it can be regarded as two-dimensional. However, it is not exactly two-dimensional because  $\mathbf{v}_s$  and  $\mathbf{H}_s$  have components along  $\mathbf{e}_\varphi$ . More precisely

$$\begin{aligned}\mathbf{v}_s &= v_{sr}(r, z, t)\mathbf{e}_r + v_{s\varphi}(r, z, t)\mathbf{e}_\varphi + v_{sz}(r, z, t)\mathbf{e}_z, \\ \mathbf{H}_s &= H_{sr}(r, z, t)\mathbf{e}_r + H_{s\varphi}(r, z, t)\mathbf{e}_\varphi + H_{sz}(r, z, t)\mathbf{e}_z, \\ \mathbf{f}_s &= f_{sr}(r, z, t)\mathbf{e}_r + f_{s\varphi}(r, z, t)\mathbf{e}_\varphi + f_{sz}(r, z, t)\mathbf{e}_z, \\ q_s &= q_s(r, z, t),\end{aligned}\tag{1.9}$$

where

$$u_r = \mathbf{u} \cdot \mathbf{e}_r, \quad u_\varphi = \mathbf{u} \cdot \mathbf{e}_\varphi, \quad u_z = \mathbf{u} \cdot \mathbf{e}_z,$$

for any  $\mathbf{u} \in \mathbb{R}^3$ . In Sect. 3, we show that a solution  $(\mathbf{v}_s, \mathbf{H}_s)$  to the following problem

$$\begin{aligned}
\mathbf{v}_{st} + (\mathbf{v}_s \cdot \nabla) \mathbf{v}_s - \nu \Delta \mathbf{v}_s + \nabla q_s - (\mathbf{H}_s \cdot \nabla) \mathbf{H}_s &= \mathbf{f}_s & \text{in } \Omega^{kT} := \Omega \times (kT, (k+1)T), \\
\operatorname{div} \mathbf{v}_s &= 0 & \text{in } \Omega^T, \\
\mathbf{H}_{st} + (\mathbf{v}_s \cdot \nabla) \mathbf{H}_s - (\mathbf{H}_s \cdot \nabla) \mathbf{v}_s - \mu \Delta \mathbf{H}_s &= \mathbf{0} & \text{in } \Omega^{kT}, \\
\operatorname{div} \mathbf{H}_s &= 0 & \text{in } \Omega^{kT}, \\
\mathbf{n} \times \operatorname{rot} \mathbf{v}_s = \mathbf{0}, \quad \mathbf{n} \cdot \mathbf{v}_s &= 0 & \text{on } S^{kT} := S \times (kT, (k+1)T), \\
\mathbf{n} \times \operatorname{rot} \mathbf{H}_s = \mathbf{0}, \quad \mathbf{n} \cdot \mathbf{H}_s &= 0 & \text{on } S^{kT}, \\
\mathbf{v}_s|_{t=kT} = \mathbf{v}_s(kT), \quad \mathbf{H}_s|_{t=kT} = \mathbf{H}_s(kT) & & \text{in } \Omega
\end{aligned} \tag{1.10}$$

is global and regular, that is we prove the following theorem:

**Theorem 1.** *Let  $\operatorname{div} \mathbf{v}_s(0) = \operatorname{div} \mathbf{H}_s(0) = 0$ . Assume that  $\operatorname{rot} \mathbf{v}_s(0), \operatorname{rot} \mathbf{H}_s(0) \in L_2(\Omega)$ ,  $\mathbf{f}_s \in L_2(\Omega^{kT})$ ,  $k \in \{0, 1, 2, \dots\}$  and*

$$\sup_k \int_{kT}^{(k+1)T} \|\mathbf{f}_s(t)\|_{L_2(\Omega)}^2 dt < +\infty.$$

*Then, there exists a weak solution to (1.10) and a constant  $A_3$  (see Lemma 3.1) such that*

$$\sup_{kT < t < (k+1)T} \|\mathbf{v}_s(t), \mathbf{H}_s(t)\|_{L_2(\Omega)}^2 + \bar{\nu} c(\Omega) \int_{kT}^{(k+1)T} \|\mathbf{v}_s(t), \mathbf{H}_s(t)\|_{H^1(\Omega)}^2 dt \leq A_3^2,$$

*where  $\bar{\nu} = \min\{\nu, \mu\}$ . If in addition  $T > 0$  is so large that  $\frac{2A_3^4}{\bar{\nu}^3} \leq T$ , then there exists an axially symmetric solution to problem (1.10) and a constant  $A_6$  (see Lemma 3.2), which depends on the initial and external data but neither on  $T$  and  $k$ , such that*

$$\mathbf{v}_s, \mathbf{H}_s \in L_\infty(kT, (k+1)T; H^1(\Omega)) \cap L_2(kT, (k+1)T; H^2(\Omega))$$

*and*

$$\sup_{kT \leq t \leq (k+1)T} \left( \|\mathbf{v}_s(t)\|_{H^1(\Omega)}^2 + \|\mathbf{H}_s(t)\|_{H^1(\Omega)}^2 \right) + \bar{\nu} \int_{kT}^{(k+1)T} \left( \|\mathbf{v}_s(t)\|_{H^2(\Omega)}^2 + \|\mathbf{H}_s(t)\|_{H^2(\Omega)}^2 \right) dt \leq A_6^2.$$

The above theorem would hold even if we take different boundary conditions for  $\mathbf{v}_s$  and  $\mathbf{H}_s$  (for a discussion about possible choices, we refer the reader to [3, 4] and the references therein). The crucial point is that (1.10) is constructed through revolving  $\Omega_{\varphi_0}$  around  $x_3$ -axis [see (1.6)], thus any solution to 2d problem in a bounded, smooth domain (for such solutions see e.g. [5, 6]), which is separated from the axis of rotation, would be a good candidate for a special solution  $(\mathbf{v}_s, \mathbf{H}_s)$ . This idea seems to work for even more general MHD models (fractional diffusion, partial diffusion, etc.), which have been studied e.g. in [7–14]. However, there are two conditions that must be met first: (a) the domain cannot contain the axis of rotation (in cited papers the whole space is considered), (b) the global estimates cannot depend on time. Although in standard approach the energy estimates do not depend on time, yet they enforce exponential decay of the external force. The method we use does not lead to exponential decay of the external data (for similar ideas see e.g. [15–18]).

In the second step, we investigate the stability of solutions to (1.10) in the class of strong solutions to (1.1). To this aim, we introduce

$$\mathbf{u} = \mathbf{v} - \mathbf{v}_s, \quad \mathbf{K} = \mathbf{H} - \mathbf{H}_s, \quad \mathbf{g} = \mathbf{f} - \mathbf{f}_s, \quad \sigma = q - q_s.$$

Then the pair  $(\mathbf{u}, \mathbf{K})$  satisfies

$$\begin{aligned}
 \mathbf{u}_t - \nu \Delta \mathbf{u} + \nabla \sigma &= -(\mathbf{u} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{v}_s - (\mathbf{v}_s \cdot \nabla) \mathbf{u} && \text{in } \Omega^{kT} := \Omega \times (kT, (k+1)T), \\
 &+ (\mathbf{K} \cdot \nabla) \mathbf{K} + (\mathbf{K} \cdot \nabla) \mathbf{H}_s + (\mathbf{H}_s \cdot \nabla) \mathbf{K} + \mathbf{g} && \\
 \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega^{kT}, \\
 \mathbf{K}_t - \mu \Delta \mathbf{K} &= -(\mathbf{u} \cdot \nabla) \mathbf{K} - (\mathbf{u} \cdot \nabla) \mathbf{H}_s - (\mathbf{v}_s \cdot \nabla) \mathbf{K} && \text{in } \Omega^{kT}, \\
 &+ (\mathbf{K} \cdot \nabla) \mathbf{u} + (\mathbf{K} \cdot \nabla) \mathbf{v}_s + (\mathbf{H}_s \cdot \nabla) \mathbf{u} && (1.11) \\
 \operatorname{div} \mathbf{K} &= 0 && \text{in } \Omega^{kT}, \\
 \mathbf{n} \times \operatorname{rot} \mathbf{u} &= \mathbf{0}, \quad \mathbf{n} \cdot \mathbf{u} = 0 && \text{on } S^{kT} := S \times (kT, (k+1)T), \\
 \mathbf{n} \times \operatorname{rot} \mathbf{K} &= \mathbf{0}, \quad \mathbf{n} \cdot \mathbf{K} = 0 && \text{on } S^{kT}, \\
 \mathbf{u}|_{t=kT} &= \mathbf{u}(kT), \quad \mathbf{K}|_{t=kT} = \mathbf{K}(kT) && \text{in } \Omega.
 \end{aligned}$$

This time, we no longer require (1.8); therefore, we expect that for the small initial and external data  $\mathbf{u}(0)$ ,  $\mathbf{K}(0)$  and  $\mathbf{g}$  the solution  $(\mathbf{u}, \mathbf{K})$  will remain small. This would imply that for  $\mathbf{v}(0)$ ,  $\mathbf{H}(0)$  and  $\mathbf{f}$  being close in suitable norms to  $\mathbf{v}_s(0)$ ,  $\mathbf{H}_s(0)$  and  $\mathbf{f}_s$ , respectively, there exists a global, unique, regular solution to (1.1) + (1.2) + (1.3) + (1.4). Now, we clearly see that  $\partial_\varphi \mathbf{v}(0)$ ,  $\partial_\varphi \mathbf{H}(0)$  and  $\partial_\varphi \mathbf{f}$  need to be small.

For solutions to (1.11), we have the following result:

**Theorem 2.** *Let the assumptions of Theorem 1 hold. Moreover, suppose that  $\operatorname{div} \mathbf{u}(0) = 0$ ,  $\operatorname{div} \mathbf{K}(0) = 0$ ,  $\operatorname{rot} \mathbf{u}(0) \in L_2(\Omega)$ ,  $\operatorname{rot} \mathbf{K}(0) \in L_2(\Omega)$  and  $\mathbf{g} \in L_2(\Omega^{kT})$  is such that*

$$\sup_k \int_{kT}^{(k+1)T} \|\mathbf{g}(t)\|_{L_{\frac{6}{5}}(\Omega)}^2 dt \leq B_1^2,$$

where  $k = 0, 1, \dots$ . If  $B_1^2$  is so small that (cf. Lemma 3.2)

$$\frac{c(\Omega)}{\bar{\nu}} B_1^2 \exp\left(\frac{c(\Omega)}{\bar{\nu}} A_6^2\right) \leq \frac{\gamma}{4},$$

time  $T > 0$  is so large that

$$T \geq \frac{2c(\Omega)}{\bar{\nu}} A_6^2 \quad \text{and} \quad \exp\left(-\frac{\bar{\nu}T}{2}\right) \leq \frac{1}{2},$$

and

$$\|\operatorname{rot} \mathbf{u}(0), \operatorname{rot} \mathbf{K}(0)\|_{L_2(\Omega)}^2 \leq \gamma,$$

where  $\gamma$  is sufficiently small number, then there exists a unique solution  $(\mathbf{u}, \mathbf{K})$  to (1.11) such that  $\mathbf{u}, \mathbf{K} \in V_2^1(\Omega^{kT})$  [see (2.2)],  $k = 0, 1, 2, \dots$  and there are two constants  $B_4$  and  $B_5$  (see Lemma 4.2) such that

$$\|\mathbf{u}, \mathbf{K}\|_{V_2^1(\Omega^{kT})}^2 \leq \gamma (B_4^2 + B_5^2).$$

In the last step, we prove the existence of global, strong solutions to (1.1) + (1.2) + (1.3) + (1.4). The main result reads:

**Theorem 3.** *Let the assumptions of Theorems 1 and 2 hold. Then, there exists a global, strong solution to (1.1) + (1.2) + (1.3) + (1.4) such that*

$$\begin{aligned}
 \mathbf{v} &= \mathbf{v}_s + \mathbf{u}, \\
 \mathbf{H} &= \mathbf{H}_s + \mathbf{K}, \\
 q &= q_s + \sigma, \\
 \mathbf{v}, \mathbf{H} &\in V_2^1(\Omega^{kT}), \\
 \nabla q &\in L_2(\Omega^{kT}),
 \end{aligned}$$

where  $k = 0, 1, 2, \dots$

For a brief description of past results concerning the regularity and existence of weak solutions, we refer the reader to the introduction in [1].

## 2. Auxiliary facts

From now on, we write  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$  and  $\Omega^{kT} = \Omega \times (kT, (k+1)T)$ . We will also frequently use

$$-\Delta = \operatorname{rot} \operatorname{rot} - \nabla \operatorname{div},$$

which suggests the following “integration by parts” formula

$$\int_{\Omega} \operatorname{rot} \operatorname{rot} \mathbf{u} \cdot \mathbf{v} d\mathbf{x} = \int_{\Omega} \operatorname{rot} \mathbf{u} \cdot \operatorname{rot} \mathbf{v} d\mathbf{x} + \int_S \mathbf{n} \times \operatorname{rot} \mathbf{u} \cdot \mathbf{v} dS.$$

All constants are generic (i.e. they may vary from line to line) and are denoted by  $c$ . Additionally, if a constant depends on the domain (e.g. in embedding inequalities), we write  $c(\Omega)$ .

Below, we introduce functions spaces and recall some technical lemmas.

### 2.1. Function spaces

By  $L_p(\Omega)$ ,  $p \in [1, \infty]$ , we denote the Lebesgue space of integrable functions. By  $H^s(\Omega)$ ,  $s \in \mathbb{N}$  and  $W_p^{2,1}(\Omega^{kT})$ ,  $p > 1$ , we denote the Sobolev spaces equipped with the following norms

$$\begin{aligned} \bullet \quad \|u\|_{H^s(\Omega)} &= \left( \sum_{|\alpha| \leq s} \int_{\Omega} |D_x^\alpha u|^2 d\mathbf{x} \right)^{\frac{1}{2}}, \\ \bullet \quad \|u\|_{W_p^{2,1}(\Omega^{kT})} &= \left( \int_{kT}^{(k+1)T} \int_{\Omega} |u_{,xx}|^p + |u_{,x}|^p + |u|^p + |u_{,t}|^p d\mathbf{x} dt \right)^{\frac{1}{p}} \end{aligned}$$

where  $D_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$ ,  $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$  and  $\alpha_i \in \mathbb{N}_0$ ,  $i = 1, 2, 3$ .

It is convenient to write

$$\|u, v\|_X^2 = \|u\|_X^2 + \|v\|_X^2, \quad (2.1)$$

where  $X$  is a Banach space.

By  $V_2^k(\Omega \times (T_1, T_2))$ , we denote a space of all functions  $u$  such that

$$\|u\|_{V_2^k(\Omega \times (T_1, T_2))} = \left( \operatorname{ess\,sup}_{t \in (T_1, T_2)} \|u(t)\|_{H^k(\Omega)}^2 + \int_{T_1}^{T_2} \|\nabla u(t)\|_{H^k(\Omega)}^2 dt \right)^{\frac{1}{2}}, \quad (2.2)$$

where  $k \in \mathbb{N}_0$ ,  $T_1 < T_2$ .

Let us now fix  $\varphi_0 \in [0, 2\pi]$  and define  $\nabla' = (\partial_r, \partial_z)$ . Since the distance between  $\Omega_{\varphi_0}$  (cf. (1.6)) and the axis of symmetry of  $\Omega$  is always positive, we may write

$$\|\mathbf{u}\|_{W_s^1(\Omega_0)} \cong \left( \int_{\Omega_0} (|\nabla' \mathbf{u}|^s + |\mathbf{u}|^s) dr dz \right)^{\frac{1}{s}}. \quad (2.3)$$

We also note that for  $\psi \in \{\mathbf{v}_s, \mathbf{H}_s\}$ , we have

$$\|\psi\|_{X(\Omega)} \leq c(\Omega_{\varphi_0}) \|\psi\|_{X(\Omega_{\varphi_0})} \leq c(\Omega) \|\psi\|_{X(\Omega)}, \quad (2.4)$$

where  $X$  is either a Lebesgue or a Sobolev space. This inequality follows immediately from the definition of  $\mathbf{v}_s, \mathbf{H}_s$  and the geometry of  $\Omega$ . More importantly, it justifies that whenever we use an embedding inequality for  $\psi$  we may take  $n = 2$ .

For function spaces defined above the following embedding will turn very useful. Namely, if  $u \in V_2^1(\Omega^{kT})$ , then  $u \in L_{10}(\Omega^{kT})$  (see [22, Lemma 3.7]) and

$$\|u\|_{L_{10}(\Omega^{kT})} \leq c(\Omega) \|u\|_{V_2^1(\Omega^t)}. \quad (2.5)$$

We will also use the interpolation inequality

$$\|\nabla u\|_{L_{\frac{5}{2}}(\Omega^{kT})} \leq c_1 \epsilon \|u\|_{W_2^{2,1}(\Omega^{kT})} + c_2 \epsilon^{-1} \|u\|_{L_2(\Omega^{kT})}. \quad (2.6)$$

## 2.2. Auxiliary results

Below, we gather crucial tools for establishing a-priori estimates for the solutions to problems (1.1), (1.10) and (1.11).

**Lemma 2.1.** (see Theorem 1.1 in [19]) *Let  $k \in \mathbb{N}_0$ . Assume that  $\Omega$  is a bounded domain such that  $\partial\Omega \in C^{k+1}$ . In addition, let  $\mathbf{F} \in H^k(\Omega)$ ,  $\operatorname{div} \mathbf{F} = 0$ . Suppose that  $\mathbf{u}$  is a solution to the following over-determined problem*

$$\begin{aligned} \operatorname{rot} \mathbf{u} &= \mathbf{F}, \\ \operatorname{div} \mathbf{u} &= 0, \\ \mathbf{u} \times \mathbf{n} &= \mathbf{0} \quad \text{or} \quad \mathbf{u} \cdot \mathbf{n} = 0. \end{aligned}$$

Then

$$\|\mathbf{u}\|_{H^{k+1}(\Omega)} \leq c(\Omega) \|\mathbf{F}\|_{H^k(\Omega)},$$

where  $k \in \mathbb{N}_0$ .

**Lemma 2.2.** *Let  $k \in \mathbb{N}_0$ ,  $\Omega \in C^k$ . Suppose that  $\mathbf{F} \in H^k(\Omega)$ ,  $\operatorname{div} \mathbf{F} = 0$ . If  $\mathbf{u}$  solves*

$$\begin{aligned} \operatorname{rot} \operatorname{rot} \mathbf{u} &= \mathbf{F} \quad \text{on } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{on } \Omega, \\ \mathbf{u} \cdot \mathbf{n} &= 0 \quad \text{on } S \\ \operatorname{rot} \mathbf{u} \times \mathbf{n} &= \mathbf{0} \quad \text{on } S, \end{aligned}$$

then

$$\|\mathbf{u}\|_{H^{k+2}(\Omega)} \leq c(k, \Omega) \|\mathbf{F}\|_{H^k(\Omega)}.$$

For the proof of the above Lemma, we refer the reader to Lemma 2.1 and problem (2.7) in [6].

**Lemma 2.3.** (cf. Lemma 3.13 in [20]) *Let us consider the Stokes problem*

$$\begin{aligned} \mathbf{v}_{,t} - \nu \Delta \mathbf{v} + \nabla p &= \mathbf{F} && \text{in } \Omega^T, \\ \operatorname{div} \mathbf{v} &= 0 && \text{in } \Omega^T, \\ \mathbf{v} \cdot \mathbf{n} &= 0 && \text{on } S^T, \\ \operatorname{rot} \mathbf{v} \times \mathbf{n} &= \mathbf{0} && \text{on } S^T, \\ \mathbf{v}|_{t=0} &= \mathbf{v}(0) && \text{on } \Omega. \end{aligned}$$

If  $\mathbf{F} \in L_s(\Omega^T)$ , where  $1 < s < \infty$ , then there exists a unique solution such that  $\mathbf{v} \in W_s^{2,1}(\Omega^T)$  and

$$\|\mathbf{v}\|_{W_s^{2,1}(\Omega^T)} + \|\nabla p\|_{L_s(\Omega^T)} \leq c(\nu, \Omega) \left( \|\mathbf{F}\|_{L_s(\Omega^T)} + \|\mathbf{v}(0)\|_{W_s^{2-\frac{2}{s}}(\Omega)} \right).$$

**Lemma 2.4.** (cf. Lemma 3.14 in [20]) *Consider the following initial-boundary value problem*

$$\begin{aligned} \mathbf{H}_{,t} - \mu \Delta \mathbf{H} &= \mathbf{G} && \text{in } \Omega^T, \\ \mathbf{H} \cdot \mathbf{n} &= 0 && \text{on } S^T, \\ \operatorname{rot} \mathbf{H} \times \mathbf{n} &= \mathbf{0} && \text{on } S^T, \\ \mathbf{H}|_{t=0} &= \mathbf{H}(0) && \text{on } \Omega. \end{aligned}$$

Assume that  $\mathbf{G} \in L_p(\Omega^T)$ , where  $1 < p < \infty$ . Then, there exists a unique solution  $\mathbf{H}$  such that  $\mathbf{H} \in W_p^{2,1}(\Omega^T)$  and

$$\|\mathbf{H}\|_{W_p^{2,1}(\Omega^T)} \leq c(\mu, \Omega) \left( \|\mathbf{G}\|_{L_p(\Omega^T)} + \|\mathbf{H}(0)\|_{W_p^{2-\frac{2}{p}}(\Omega)} \right).$$

**Lemma 2.5.** (Agmon inequalities; cf. (1.2.44) in [21]) *Let  $\Omega \subset \mathbb{R}^n$ ,  $\partial\Omega \in \mathcal{C}^n$ . If  $\varphi \in H^2(\Omega)$ , then*

$$\begin{aligned} \|\varphi\|_{L_\infty(\Omega)} &\leq c(n, \Omega) \|\varphi\|_{L_2(\Omega)}^{\frac{1}{2}} \|\varphi\|_{H^2(\Omega)}^{\frac{1}{2}} && n = 2, \\ \|\varphi\|_{L_\infty(\Omega)} &\leq c(n, \Omega) \|\varphi\|_{H^1(\Omega)}^{\frac{1}{2}} \|\varphi\|_{H^2(\Omega)}^{\frac{1}{2}} && n = 3. \end{aligned}$$

### 3. The existence and properties of solutions to (1.10)

Using energy methods, we prove a priori estimates for a solution  $(\mathbf{v}_s, \mathbf{H}_s)$  to (1.10). Therefore, the existence of the solution will follow from the Faedo–Galerkin method. We start with the basic global energy estimate.

**Lemma 3.1.** *Let  $(\mathbf{v}_s, \mathbf{H}_s)$  be a solution to (1.10),  $\operatorname{div} \mathbf{v}_s(0) = 0$ ,  $\operatorname{div} \mathbf{H}_s(0) = 0$ . Suppose that  $\bar{\nu} = \min\{\nu, \mu\}$ ,  $k \in \mathbb{N}_0$ ,*

$$\begin{aligned} A_1^2 &\equiv \frac{c(\Omega)}{\bar{\nu}} \sup_{k \in \mathbb{N}_0} \int_{kT}^{(k+1)T} \|\mathbf{f}_s(t)\|_{L_2(\Omega)}^2 dt < \infty, \\ A_2^2 &\equiv \frac{A_1^2}{1 - e^{-\bar{\nu}c(\Omega)T}} + \|\mathbf{v}_s(0), \mathbf{H}_s(0)\|_{L_2(\Omega)}^2 e^{-\bar{\nu}c(\Omega)T} < \infty. \end{aligned} \tag{3.1}$$

Then

$$\begin{aligned} \|\mathbf{v}_s(kT), \mathbf{H}_s(kT)\|_{L_2(\Omega)}^2 &\leq A_2^2, \\ \|\mathbf{v}_s(t), \mathbf{H}_s(t)\|_{L_2(\Omega)}^2 + \bar{\nu}c(\Omega) \int_{kT}^t \|\mathbf{v}_s(\tau), \mathbf{H}_s(\tau)\|_{H^1(\Omega)}^2 d\tau &\leq A_1^2 + A_2^2 \equiv A_3^2, \end{aligned} \tag{3.2}$$

where  $t \in (kT, (k+1)T]$ .

*Proof.* Multiplying (1.10)<sub>1,3</sub> by  $\mathbf{v}_s$  and  $\mathbf{H}_s$ , respectively, integrating over  $\Omega$ , using (1.10)<sub>2,4</sub> and the boundary conditions (1.10)<sub>5,6</sub>, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{v}_s, \mathbf{H}_s\|_{L_2(\Omega)}^2 + \nu \|\operatorname{rot} \mathbf{v}_s\|_{L_2(\Omega)}^2 + \mu \|\operatorname{rot} \mathbf{H}_s\|_{L_2(\Omega)}^2 = \int_{\Omega} \mathbf{f}_s \cdot \mathbf{v}_s dx.$$

Utilizing Lemma 2.1 and the Hölder and Young inequalities, we get

$$\frac{d}{dt} \|\mathbf{v}_s, \mathbf{H}_s\|_{L_2(\Omega)}^2 + \bar{\nu} c(\Omega) \|\mathbf{v}_s, \mathbf{H}_s\|_{L_2(\Omega)}^2 \leq \frac{1}{\bar{\nu} c(\Omega)} \|\mathbf{f}_s\|_{L_2(\Omega)}^2, \quad (3.3)$$

which implies

$$\frac{d}{dt} \left( \|\mathbf{v}_s, \mathbf{H}_s\|_{L_2(\Omega)}^2 e^{\bar{\nu} c(\Omega)t} \right) \leq \frac{1}{\bar{\nu} c(\Omega)} \|\mathbf{f}_s\|_{L_2(\Omega)}^2 e^{\bar{\nu} c(\Omega)t}.$$

Integrating the above inequality with respect to time from  $t = kT$  to  $t \in (kT, (k+1)T]$  yields

$$\|\mathbf{v}_s(t), \mathbf{H}_s(t)\|_{L_2(\Omega)}^2 \leq \frac{1}{\bar{\nu} c(\Omega)} \int_{kT}^t \|\mathbf{f}_s(\tau)\|_{L_2(\Omega)}^2 d\tau + e^{-\bar{\nu} c(\Omega)(t-kT)} \|\mathbf{v}_s(kT), \mathbf{H}_s(kT)\|_{L_2(\Omega)}^2.$$

Setting  $t = (k+1)T$  gives

$$\|\mathbf{v}_s((k+1)T), \mathbf{H}_s((k+1)T)\|_{L_2(\Omega)}^2 \leq \frac{1}{\bar{\nu} c(\Omega)} \int_{kT}^{(k+1)T} \|\mathbf{f}_s(t)\|_{L_2(\Omega)}^2 dt + e^{-\bar{\nu} c(\Omega)T} \|\mathbf{v}_s(kT), \mathbf{H}_s(kT)\|_{L_2(\Omega)}^2.$$

Iterating the above procedure, we get

$$\|\mathbf{v}_s(kT), \mathbf{H}_s(kT)\|_{L_2(\Omega)}^2 \leq \frac{A_1^2}{1 - e^{-\bar{\nu} c(\Omega)T}} + e^{-\bar{\nu} c(\Omega)T} \|\mathbf{v}_s(0), \mathbf{H}_s(0)\|_{L_2(\Omega)}^2,$$

which proves (3.2)<sub>1</sub>. To conclude (3.2)<sub>2</sub>, we integrate (3.3) with respect to time and use the above inequality. This ends the proof.  $\square$

In the below lemma, we establish higher-order estimates for weak solutions to (1.10).

**Lemma 3.2.** *Let the assumptions of Lemma 3.1 hold. Assume that  $T > 0$  is so large that  $\frac{2A_3^4}{\bar{\nu}^3} \leq T$ . Let*

$$\begin{aligned} A_4^2 &\equiv A_1^2 \exp\left(\frac{A_3^4}{\bar{\nu}^2}\right), \\ A_5^2 &\equiv \frac{A_4^2}{1 - \exp\left(-\frac{\bar{\nu}T}{2}\right)} + \|\operatorname{rot} \mathbf{v}_s(0), \operatorname{rot} \mathbf{H}_s(0)\|_{L_2(\Omega)}^2 \exp\left(-\frac{\bar{\nu}T}{2}\right) < +\infty, \\ A_6^2 &\equiv A_5^2 \frac{A_3^4}{\bar{\nu}^2} + A_1^2 + A_5^2. \end{aligned}$$

Then

$$\begin{aligned} \|\operatorname{rot} \mathbf{v}_s(kT), \operatorname{rot} \mathbf{H}_s(kT)\|_{L_2(\Omega)}^2 &\leq A_5^2, \\ \|\operatorname{rot} \mathbf{v}_s(t), \operatorname{rot} \mathbf{H}_s(t)\|_{L_2(\Omega)}^2 + \bar{\nu} \int_{kT}^t \|\operatorname{rot}^2 \mathbf{v}_s(\tau), \operatorname{rot}^2 \mathbf{H}_s(\tau)\|_{L_2(\Omega)}^2 d\tau &\leq A_6^2, \end{aligned} \quad (3.4)$$

where  $t \in (kT, (k+1)T]$ ,  $k \in \mathbb{N}_0$ .

*Proof.* We begin with multiplying (1.10)<sub>1,3</sub> by  $\operatorname{rot}^2 \mathbf{v}_s$  and  $\operatorname{rot}^2 \mathbf{H}_s$ , respectively, integrating the result over  $\Omega$  and using the boundary conditions (1.10)<sub>5,6</sub>

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\operatorname{rot} \mathbf{v}_s, \operatorname{rot} \mathbf{H}_s\|_{L_2(\Omega)}^2 + \bar{\nu} \|\operatorname{rot}^2 \mathbf{v}_s, \operatorname{rot}^2 \mathbf{H}_s\|_{L_2(\Omega)}^2 \\ &\leq - \int_{\Omega} ((\mathbf{v}_s \cdot \nabla) \mathbf{v}_s - (\mathbf{H}_s \cdot \nabla) \mathbf{H}_s) \cdot \operatorname{rot}^2 \mathbf{v}_s dx \\ &\quad - \int_{\Omega} ((\mathbf{v}_s \cdot \nabla) \mathbf{H}_s - (\mathbf{H}_s \cdot \nabla) \mathbf{v}_s) \cdot \operatorname{rot}^2 \mathbf{H}_s dx + \int_{\Omega} \mathbf{f}_s \cdot \operatorname{rot}^2 \mathbf{v}_s dx. \end{aligned}$$



Utilizing the Hölder and Young inequalities and Lemma 2.5, we get

$$\begin{aligned}
-\int_{\Omega} ((\mathbf{v}_s \cdot \nabla) \mathbf{v}_s) \cdot \operatorname{rot}^2 \mathbf{v}_s dx &\leq \|\operatorname{rot}^2 \mathbf{v}_s\|_{L_2(\Omega)} \|\nabla \mathbf{v}_s\|_{L_2(\Omega)} \|\mathbf{v}_s\|_{L_{\infty}(\Omega)} \\
&\leq c(\Omega) \|\mathbf{v}_s\|_{H^2(\Omega)}^{\frac{3}{2}} \|\nabla \mathbf{v}_s\|_{L_2(\Omega)} \|\mathbf{v}_s\|_{L_2(\Omega)}^{\frac{1}{2}} \\
&\leq \epsilon_1 \|\mathbf{v}_s\|_{H^2(\Omega)}^2 + c\left(\frac{1}{\epsilon_1}, \Omega\right) \|\mathbf{v}_s\|_{H^1(\Omega)}^4 \|\mathbf{v}_s\|_{L_2(\Omega)}^2.
\end{aligned}$$

Analogously

$$\begin{aligned}
\int_{\Omega} ((\mathbf{H}_s \cdot \nabla) \mathbf{H}_s) \cdot \operatorname{rot}^2 \mathbf{v}_s dx &\leq \|\operatorname{rot}^2 \mathbf{v}_s\|_{L_2(\Omega)} \|\nabla \mathbf{H}_s\|_{L_2(\Omega)} \|\mathbf{H}_s\|_{L_{\infty}(\Omega)} \\
&\leq c(\Omega) \|\operatorname{rot}^2 \mathbf{v}_s\|_{L_2(\Omega)} \|\mathbf{H}_s\|_{H^2(\Omega)}^{\frac{1}{2}} \|\nabla \mathbf{H}_s\|_{L_2(\Omega)} \|\mathbf{H}_s\|_{L_2(\Omega)}^{\frac{1}{2}} \\
&\leq \epsilon_2 \|\operatorname{rot}^2 \mathbf{v}_s\|_{L_2(\Omega)}^2 + \epsilon_3 \|\mathbf{H}_s\|_{H^2(\Omega)}^2 + c\left(\frac{1}{\epsilon_2}, \frac{1}{\epsilon_3}, \Omega\right) \|\mathbf{H}_s\|_{H^1(\Omega)}^4 \|\mathbf{H}_s\|_{L_2(\Omega)}^2, \\
-\int_{\Omega} ((\mathbf{v}_s \cdot \nabla) \mathbf{H}_s) \cdot \operatorname{rot}^2 \mathbf{H}_s dx &\leq \|\operatorname{rot}^2 \mathbf{H}_s\|_{L_2(\Omega)} \|\nabla \mathbf{H}_s\|_{L_2(\Omega)} \|\mathbf{v}_s\|_{L_{\infty}(\Omega)} \\
&\leq c(\Omega) \|\operatorname{rot}^2 \mathbf{H}_s\|_{L_2(\Omega)} \|\mathbf{v}_s\|_{H^2(\Omega)}^{\frac{1}{2}} \|\nabla \mathbf{H}_s\|_{L_2(\Omega)} \|\mathbf{v}_s\|_{L_2(\Omega)}^{\frac{1}{2}} \\
&\leq \epsilon_4 \|\operatorname{rot}^2 \mathbf{H}_s\|_{L_2(\Omega)}^2 + \epsilon_5 \|\mathbf{v}_s\|_{H^2(\Omega)}^2 + c\left(\frac{1}{\epsilon_4}, \frac{1}{\epsilon_5}, \Omega\right) \|\mathbf{H}_s\|_{H^1(\Omega)}^4 \|\mathbf{v}_s\|_{L_2(\Omega)}^2,
\end{aligned}$$

and

$$\begin{aligned}
\int_{\Omega} ((\mathbf{H}_s \cdot \nabla) \mathbf{v}_s) \cdot \operatorname{rot}^2 \mathbf{H}_s dx &\leq \|\operatorname{rot}^2 \mathbf{H}_s\|_{L_2(\Omega)} \|\nabla \mathbf{v}_s\|_{L_2(\Omega)} \|\mathbf{H}_s\|_{L_{\infty}(\Omega)} \\
&\leq c(\Omega) \|\mathbf{H}_s\|_{H^2(\Omega)}^{\frac{3}{2}} \|\nabla \mathbf{v}_s\|_{L_2(\Omega)} \|\mathbf{H}_s\|_{L_2(\Omega)}^{\frac{1}{2}} \\
&\leq \epsilon_6 \|\mathbf{H}_s\|_{H^2(\Omega)}^2 + c\left(\frac{1}{\epsilon_6}, \Omega\right) \|\mathbf{v}_s\|_{H^1(\Omega)}^4 \|\mathbf{H}_s\|_{L_2(\Omega)}^2.
\end{aligned}$$

Using Lemma 2.2 and taking  $\epsilon_1, \dots, \epsilon_6$  sufficiently small, we obtain

$$\begin{aligned}
&\frac{d}{dt} \|\operatorname{rot} \mathbf{v}_s, \operatorname{rot} \mathbf{H}_s\|_{L_2(\Omega)}^2 + \bar{\nu} \|\operatorname{rot}^2 \mathbf{v}_s, \operatorname{rot}^2 \mathbf{H}_s\|_{L_2(\Omega)}^2 \\
&\leq \frac{c(\Omega)}{\bar{\nu}} \|\mathbf{v}_s, \mathbf{H}_s\|_{H^1(\Omega)}^4 \|\mathbf{v}_s, \mathbf{H}_s\|_{L_2(\Omega)}^2 + \frac{1}{\bar{\nu}} \|\mathbf{f}_s\|_{L_2(\Omega)}^2.
\end{aligned} \tag{3.5}$$

In light of Lemma 2.1, we have

$$\begin{aligned}
&\frac{d}{dt} \|\operatorname{rot} \mathbf{v}_s, \operatorname{rot} \mathbf{H}_s\|_{L_2(\Omega)}^2 + \bar{\nu} \|\operatorname{rot} \mathbf{v}_s, \operatorname{rot} \mathbf{H}_s\|_{L_2(\Omega)}^2 \\
&\leq \frac{c(\Omega)}{\bar{\nu}} \|\mathbf{v}_s, \mathbf{H}_s\|_{H^1(\Omega)}^2 \|\mathbf{v}_s, \mathbf{H}_s\|_{L_2(\Omega)}^2 \|\operatorname{rot} \mathbf{v}_s, \operatorname{rot} \mathbf{H}_s\|_{L_2(\Omega)}^2 + \frac{c(\Omega)}{\bar{\nu}} \|\mathbf{f}_s\|_{L_2(\Omega)}^2.
\end{aligned}$$

From the above inequality it follows that

$$\begin{aligned} & \frac{d}{dt} \left( \|\operatorname{rot} \mathbf{v}_s, \operatorname{rot} \mathbf{H}_s\|_{L_2(\Omega)}^2 \exp \left( \bar{\nu} t - \frac{c(\Omega)}{\bar{\nu}} \int_{kT}^t \|\mathbf{v}_s(\tau), \mathbf{H}_s(\tau)\|_{H^1(\Omega)}^2 \|\mathbf{v}_s(\tau), \mathbf{H}_s(\tau)\|_{L_2(\Omega)}^2 d\tau \right) \right) \\ & \leq \frac{c(\Omega)}{\bar{\nu}} \|\mathbf{f}_s\|_{L_2(\Omega)}^2 \exp \left( \bar{\nu} t - \frac{c(\Omega)}{\bar{\nu}} \int_{kT}^t \|\mathbf{v}_s(\tau), \mathbf{H}_s(\tau)\|_{H^1(\Omega)}^2 \|\mathbf{v}_s(\tau), \mathbf{H}_s(\tau)\|_{L_2(\Omega)}^2 d\tau \right). \end{aligned}$$

Integrating with respect to time from  $t = kT$  to  $t \in (kT; (k+1)T)$  yields

$$\begin{aligned} \|\operatorname{rot} \mathbf{v}_s(t), \operatorname{rot} \mathbf{H}_s(t)\|_{L_2(\Omega)}^2 & \leq \frac{c(\Omega)}{\bar{\nu}} \int_{kT}^t \|\mathbf{f}_s(\tau)\|_{L_2(\Omega)}^2 d\tau \exp \left( \frac{c(\Omega)}{\bar{\nu}} \int_{kT}^t \|\mathbf{v}_s(\tau), \mathbf{H}_s(\tau)\|_{H^1(\Omega)}^2 \|\mathbf{v}_s(\tau), \mathbf{H}_s(\tau)\|_{L_2(\Omega)}^2 d\tau \right) \\ & + \|\operatorname{rot} \mathbf{v}_s(kT), \operatorname{rot} \mathbf{H}_s(kT)\|_{L_2(\Omega)}^2 \exp \left( -\bar{\nu}(t - kT) + \frac{c(\Omega)}{\bar{\nu}} \int_{kT}^t \|\mathbf{v}_s(\tau), \mathbf{H}_s(\tau)\|_{H^1(\Omega)}^2 \|\mathbf{v}_s(\tau), \mathbf{H}_s(\tau)\|_{L_2(\Omega)}^2 d\tau \right). \end{aligned}$$

From Lemma 3.1 it follows that

$$\sup_{k \in \mathbb{N}_0} \frac{c(\Omega)}{\bar{\nu}} \int_{kT}^t \|\mathbf{v}_s(\tau), \mathbf{H}_s(\tau)\|_{H^1(\Omega)}^2 \|\mathbf{v}_s(\tau), \mathbf{H}_s(\tau)\|_{L_2(\Omega)}^2 d\tau \leq \frac{A_3^4}{\bar{\nu}^2},$$

thus

$$\begin{aligned} \|\operatorname{rot} \mathbf{v}_s(t), \operatorname{rot} \mathbf{H}_s(t)\|_{L_2(\Omega)}^2 & \leq \frac{c(\Omega)}{\bar{\nu}} \int_{kT}^t \|\mathbf{f}_s(\tau)\|_{L_2(\Omega)}^2 d\tau \exp \left( \frac{A_3^4}{\bar{\nu}^2} \right) \\ & + \|\operatorname{rot} \mathbf{v}_s(kT), \operatorname{rot} \mathbf{H}_s(kT)\|_{L_2(\Omega)}^2 \exp \left( -\bar{\nu}(t - kT) + \frac{A_3^4}{\bar{\nu}^2} \right). \end{aligned}$$

Setting  $t = (k+1)T$  and using that  $T \geq \frac{2A_3^4}{\bar{\nu}^3}$  we have

$$\|\operatorname{rot} \mathbf{v}_s((k+1)T), \operatorname{rot} \mathbf{H}_s((k+1)T)\|_{L_2(\Omega)}^2 \leq A_4^2 + \|\operatorname{rot} \mathbf{v}_s(kT), \operatorname{rot} \mathbf{H}_s(kT)\|_{L_2(\Omega)}^2 \exp \left( -\frac{\bar{\nu}T}{2} \right).$$

Iterating the above procedure yields

$$\|\operatorname{rot} \mathbf{v}_s(kT), \operatorname{rot} \mathbf{H}_s(kT)\|_{L_2(\Omega)}^2 \leq \frac{A_4^2}{1 - \exp \left( -\frac{\bar{\nu}T}{2} \right)} + \|\operatorname{rot} \mathbf{v}_s(0), \operatorname{rot} \mathbf{H}_s(0)\|_{L_2(\Omega)}^2 \exp \left( -\frac{\bar{\nu}T}{2} \right),$$

which proves (3.4)<sub>1</sub>.

To prove (3.4)<sub>2</sub>, we integrate (3.5) with respect to time from  $t = kT$  to  $t \in (kT, (k+1)T]$  and use Lemma 2.1. Then

$$\begin{aligned} & \|\operatorname{rot} \mathbf{v}_s(t), \operatorname{rot} \mathbf{H}_s(t)\|_{L_2(\Omega)}^2 + \bar{\nu} \int_{kT}^t \|\operatorname{rot}^2 \mathbf{v}_s(\tau), \operatorname{rot}^2 \mathbf{H}_s(\tau)\|_{L_2(\Omega)}^2 d\tau \\ & \leq \frac{c(\Omega)}{\bar{\nu}} \sup_{kT \leq t \leq (k+1)T} \|\operatorname{rot} \mathbf{v}_s(t), \operatorname{rot} \mathbf{H}_s(t)\|_{L_2(\Omega)}^2 \int_{kT}^t \|\mathbf{v}_s(\tau), \mathbf{H}_s(\tau)\|_{H^1(\Omega)}^2 \|\mathbf{v}_s(\tau), \mathbf{H}_s(\tau)\|_{L_2(\Omega)}^2 d\tau \\ & + \frac{c(\Omega)}{\bar{\nu}} \int_{kT}^t \|\mathbf{f}_s(\tau)\|_{L_2(\Omega)}^2 d\tau + \|\operatorname{rot} \mathbf{v}_s(kT), \operatorname{rot} \mathbf{H}_s(kT)\|_{L_2(\Omega)}^2 \leq A_5^2 \frac{A_3^4}{\bar{\nu}^2} + A_1^2 + A_5^2. \end{aligned}$$

This completes the proof.  $\square$

**Remark 3.3.** Under the assumptions of Lemma 3.2, we have

$$\|\mathbf{v}_s\|_{W_2^{2,1}(\Omega^{kT})} + \|\mathbf{H}_s\|_{W_2^{2,1}(\Omega^{kT})} + \|\nabla p\|_{L_2(\Omega^{kT})} \leq c(\Omega) (A_6^2 + A_3 A_6 + A_1 + A_5) \equiv A_7. \quad (3.6)$$

Indeed, in light of Lemmas 2.3 and 2.4, we have

$$\begin{aligned} & \|\mathbf{v}_s\|_{W_2^{2,1}(\Omega^{kT})} + \|\mathbf{H}_s\|_{W_2^{2,1}(\Omega^{kT})} + \|\nabla p\|_{L_2(\Omega^{kT})} \leq \|(\mathbf{v}_s \cdot \nabla) \mathbf{v}_s\|_{L_2(\Omega^{kT})} + \|(\mathbf{H}_s \cdot \nabla) \mathbf{H}_s\|_{L_2(\Omega^{kT})} \\ & + \|(\mathbf{v}_s \cdot \nabla) \mathbf{H}_s\|_{L_2(\Omega^{kT})} + \|(\mathbf{H}_s \cdot \nabla) \mathbf{v}_s\|_{L_2(\Omega^{kT})} + \|\mathbf{f}_s\|_{L_2(\Omega^{kT})} + \|\mathbf{v}_s(kT)\|_{H^1(\Omega)} + \|\mathbf{H}_s(kT)\|_{H^1(\Omega)}. \end{aligned}$$

Using the Hölder inequality, we get

$$\begin{aligned} & \|(\mathbf{v}_s \cdot \nabla) \mathbf{v}_s\|_{L_2(\Omega^{kT})} + \|(\mathbf{H}_s \cdot \nabla) \mathbf{H}_s\|_{L_2(\Omega^{kT})} + \|(\mathbf{v}_s \cdot \nabla) \mathbf{H}_s\|_{L_2(\Omega^{kT})} + \|(\mathbf{H}_s \cdot \nabla) \mathbf{v}_s\|_{L_2(\Omega^{kT})} \\ & \leq \left( \|\mathbf{v}_s\|_{L_{10}(\Omega^{kT})} + \|\mathbf{H}_s\|_{L_{10}(\Omega^{kT})} \right) \left( \|\nabla \mathbf{v}_s\|_{L_{\frac{5}{2}}(\Omega^{kT})} + \|\nabla \mathbf{H}_s\|_{L_{\frac{5}{2}}(\Omega^{kT})} \right). \end{aligned}$$

From Lemma 3.2 it follows that  $\|\mathbf{v}_s\|_{V_2^1(\Omega^{kT})}^2 + \|\mathbf{H}_s\|_{V_2^1(\Omega^{kT})}^2 \leq A_6^2$ . Combining that with (2.5) yields

$$\|\mathbf{v}_s\|_{L_{10}(\Omega^{kT})} + \|\mathbf{H}_s\|_{L_{10}(\Omega^{kT})} \leq c(\Omega) A_6.$$

By (2.6), we have

$$\begin{aligned} \|\nabla \mathbf{v}_s\|_{L_{\frac{5}{2}}(\Omega^{kT})} + \|\nabla \mathbf{H}_s\|_{L_{\frac{5}{2}}(\Omega^{kT})} & \leq c_1 \epsilon^{\frac{1}{2}} \left( \|\mathbf{v}_s\|_{W_2^{2,1}(\Omega^{kT})} + \|\mathbf{H}_s\|_{W_2^{2,1}(\Omega^{kT})} \right) \\ & + c_2 \epsilon^{-\frac{1}{2}} \left( \|\mathbf{v}_s\|_{L_2(\Omega^{kT})} + \|\mathbf{H}_s\|_{L_2(\Omega^{kT})} \right). \end{aligned}$$

Eventually

$$\|(\mathbf{v}_s \cdot \nabla) \mathbf{v}_s\|_{L_2(\Omega^{kT})} + \|(\mathbf{H}_s \cdot \nabla) \mathbf{H}_s\|_{L_2(\Omega^{kT})} + \|(\mathbf{v}_s \cdot \nabla) \mathbf{H}_s\|_{L_2(\Omega^{kT})} + \|(\mathbf{H}_s \cdot \nabla) \mathbf{v}_s\|_{L_2(\Omega^{kT})} \leq c(\Omega) A_6^2 + A_3 A_6.$$

The above estimate with the estimates from Lemmas 3.1, 3.2 along with Lemma 2.1 ends this remark.

*Proof of Theorem 1.* The proof is straightforward and follows from the energy estimates (see Lemmas 3.1, 3.2) and the Galerkin method. As the basis functions, we can take the eigenfunctions of the Laplacian equipped with the Navier boundary conditions (cf. Sections 2.3 and 3.2 in [6] and Section 3 in [23]).  $\square$

#### 4. Stability of solutions to (1.10)

In this section, we examine the stability of solutions to (1.10) in the class of solutions to (1.1) + (1.2) + (1.3) + (1.4). The key point is the analysis of solutions to (1.11).

**Lemma 4.1.** *Let the assumptions of Lemma 3.2 hold. In addition suppose that  $T > 0$  is so large that  $T \geq \frac{2c(\Omega)}{\bar{\nu}} A_6^2$  and  $\exp\left(-\frac{\bar{\nu}T}{2}\right) \leq \frac{1}{2}$ , where  $A_6$  was introduced in (3.4).*

*Let  $\mathbf{g} \in L_2(kT, (k+1)T; L_{\frac{6}{5}}(\Omega))$  satisfy*

$$\sup_{k \in \mathbb{N}_0} \int_{kT}^{(k+1)T} \|\mathbf{g}(t)\|_{L_{\frac{6}{5}}(\Omega)}^2 dt \leq B_1^2.$$

*Suppose that  $B_1^2$  is so small that*

$$\frac{c(\Omega)}{\bar{\nu}} B_1^2 \exp\left(\frac{c(\Omega)}{\bar{\nu}} A_6^2\right) \leq \frac{\gamma}{4}.$$

If  $\|\mathbf{u}(0), \mathbf{K}(0)\|_{L_2(\Omega)}^2 \leq \gamma$ , then

$$\begin{aligned}
 & \bullet \quad \|\mathbf{u}(kT), \mathbf{K}(kT)\|_{L_2(\Omega)}^2 \leq \gamma, \\
 & \bullet \quad \sup_{kT \leq t \leq (k+1)T} \|\mathbf{u}(t), \mathbf{K}(t)\|_{L_2(\Omega)}^2 \leq \gamma \exp\left(\frac{c(\Omega)}{\bar{\nu}} A_6^2\right) \equiv \gamma B_2^2 \\
 & \bullet \quad \int_{kT}^{(k+1)T} \|\operatorname{rot} \mathbf{u}(\tau), \operatorname{rot} \mathbf{K}(\tau)\|_{L_2(\Omega)}^2 d\tau \leq \gamma \frac{c(\Omega)}{\bar{\nu}} \exp\left(\frac{c(\Omega)}{\bar{\nu}} A_6^2\right) A_6^2 + \frac{\gamma}{4} + \gamma \equiv \gamma B_3^2.
 \end{aligned} \tag{4.1}$$

for  $k \in \mathbb{N}_0$ .

*Proof.* We multiply (1.11)<sub>1,3</sub> by  $\mathbf{u}$  and  $\mathbf{K}$ , respectively, integrate the result over  $\Omega$  and use the boundary conditions (1.11)<sub>5,6</sub>

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \|\mathbf{u}, \mathbf{K}\|_{L_2(\Omega)}^2 + \nu \|\operatorname{rot} \mathbf{u}\|_{L_2(\Omega)}^2 + \mu \|\operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^2 = - \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{u} dx - \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{v}_s \cdot \mathbf{u} dx - \int_{\Omega} (\mathbf{v}_s \cdot \nabla) \mathbf{u} \cdot \mathbf{u} dx \\
 & + \int_{\Omega} (\mathbf{K} \cdot \nabla) \mathbf{K} \cdot \mathbf{u} dx + \int_{\Omega} (\mathbf{K} \cdot \nabla) \mathbf{H}_s \cdot \mathbf{u} dx + \int_{\Omega} (\mathbf{H}_s \cdot \nabla) \mathbf{K} \cdot \mathbf{u} dx + \int_{\Omega} \mathbf{g} \cdot \mathbf{u} dx - \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{K} \cdot \mathbf{K} dx \\
 & - \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{H}_s \cdot \mathbf{K} dx - \int_{\Omega} (\mathbf{v}_s \cdot \nabla) \mathbf{K} \cdot \mathbf{K} dx + \int_{\Omega} (\mathbf{K} \cdot \nabla) \mathbf{u} \cdot \mathbf{K} + \int_{\Omega} (\mathbf{K} \cdot \nabla) \mathbf{v}_s \cdot \mathbf{K} dx + \int_{\Omega} (\mathbf{H}_s \cdot \nabla) \mathbf{u} \cdot \mathbf{K} dx =: \sum_{k=1}^{13} J_k.
 \end{aligned}$$

We easily note that  $J_1, J_3, J_8, J_{10}$  vanish. We also have  $J_4 = -J_{11}$  and  $J_6 = -J_{13}$ . By the Hölder and Young inequalities

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \|\mathbf{u}, \mathbf{K}\|_{L_2(\Omega)}^2 + \bar{\nu} \|\operatorname{rot} \mathbf{u}, \operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^2 \leq \epsilon_2 \|\mathbf{u}\|_{L_6(\Omega)}^2 + \frac{1}{4\epsilon_2} \|\mathbf{u}\|_{L_2(\Omega)}^2 \|\nabla \mathbf{v}_s\|_{L_3(\Omega)}^2 \\
 & + \epsilon_5 \|\mathbf{K}\|_{L_6(\Omega)}^2 + \frac{1}{4\epsilon_5} \|\mathbf{u}\|_{L_2(\Omega)}^2 \|\nabla \mathbf{H}_s\|_{L_3(\Omega)}^2 + \epsilon_7 \|\mathbf{u}\|_{L_6(\Omega)}^2 + \frac{1}{4\epsilon_7} \|\mathbf{g}\|_{L_{\frac{6}{5}}(\Omega)}^2 \\
 & + \epsilon_9 \|\mathbf{u}\|_{L_6(\Omega)}^2 + \frac{1}{4\epsilon_9} \|\mathbf{K}\|_{L_2(\Omega)}^2 \|\nabla \mathbf{H}_s\|_{L_3(\Omega)}^2 + \epsilon_{12} \|\mathbf{K}\|_{L_6(\Omega)}^2 + \frac{1}{4\epsilon_{12}} \|\mathbf{K}\|_{L_2(\Omega)}^2 \|\nabla \mathbf{v}_s\|_{L_3(\Omega)}^2.
 \end{aligned}$$

Choosing  $\epsilon_2, \epsilon_5, \epsilon_7, \epsilon_9$  and  $\epsilon_{12}$  sufficiently small and using Lemma 2.1 to estimate  $L_6$ -norms, we get

$$\frac{d}{dt} \|\mathbf{u}, \mathbf{K}\|_{L_2(\Omega)}^2 + \bar{\nu} \|\operatorname{rot} \mathbf{u}, \operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^2 \leq \frac{c(\Omega)}{\bar{\nu}} \|\mathbf{u}, \mathbf{K}\|_{L_2(\Omega)}^2 \|\nabla \mathbf{v}_s, \nabla \mathbf{H}_s\|_{L_3(\Omega)}^2 + \frac{c(\Omega)}{\bar{\nu}} \|\mathbf{g}\|_{L_{\frac{6}{5}}(\Omega)}^2. \tag{4.2}$$

Applying Lemma 2.1 for the second term on the left-hand side, we obtain

$$\begin{aligned}
 & \frac{d}{dt} \left( \|\mathbf{u}, \mathbf{K}\|_{L_2(\Omega)}^2 \exp \left( \bar{\nu} c(\Omega) t - \frac{c(\Omega)}{\bar{\nu}} \int_{kT}^t \|\nabla \mathbf{v}_s(\tau), \nabla \mathbf{H}_s(\tau)\|_{L_3(\Omega)}^2 d\tau \right) \right) \\
 & \leq \frac{c(\Omega)}{\bar{\nu}} \|\mathbf{g}\|_{L_{\frac{6}{5}}(\Omega)}^2 \exp \left( \bar{\nu} c(\Omega) t - \frac{c(\Omega)}{\bar{\nu}} \int_{kT}^t \|\nabla \mathbf{v}_s(\tau), \nabla \mathbf{H}_s(\tau)\|_{L_3(\Omega)}^2 d\tau \right).
 \end{aligned}$$

Integration with respect to time from  $t = kT$  to  $t \in (kT, (k+1)T)$  yields

$$\begin{aligned} \|\mathbf{u}(t), \mathbf{K}(t)\|_{L_2(\Omega)}^2 &\leq \frac{c(\Omega)}{\bar{\nu}} \int_{kT}^t \|\mathbf{g}(\tau)\|_{L_{\frac{6}{5}}(\Omega)}^2 \exp \left( \bar{\nu} c(\Omega) \tau - \frac{c(\Omega)}{\bar{\nu}} \int_{kT}^{\tau} \|\nabla \mathbf{v}_s(t'), \nabla \mathbf{H}_s(t')\|_{L_3(\Omega)}^2 dt' \right) d\tau \\ &\cdot \exp \left( -\bar{\nu} c(\Omega) t + \frac{c(\Omega)}{\bar{\nu}} \int_{kT}^t \|\nabla \mathbf{v}_s(\tau), \nabla \mathbf{H}_s(\tau)\|_{L_3(\Omega)}^2 d\tau \right) \\ &+ \|\mathbf{u}(kT), \mathbf{K}(kT)\|_{L_2(\Omega)}^2 \exp \left( -\bar{\nu} c(\Omega)(t - kT) + \frac{c(\Omega)}{\bar{\nu}} \int_{kT}^t \|\nabla \mathbf{v}_s(\tau), \nabla \mathbf{H}_s(\tau)\|_{L_3(\Omega)}^2 d\tau \right). \end{aligned} \quad (4.3)$$

By the Hölder inequality, Sobolev embedding and Lemmas 2.2 and 3.2, we have

$$\sup_{k \in \mathbb{N}_0} \int_{kT}^t \|\nabla \mathbf{v}_s(\tau), \nabla \mathbf{H}_s(\tau)\|_{L_3(\Omega)}^2 d\tau \leq c(\Omega) A_6^2, \quad t \in (kT, (k+1)T).$$

We take  $t = (k+1)T$  and use that  $T \geq \frac{2c(\Omega)}{\bar{\nu}} A_6^2$

$$\|\mathbf{u}((k+1)T), \mathbf{K}((k+1)T)\|_{L_2(\Omega)}^2 \leq \frac{c(\Omega)}{\bar{\nu}} B_1^2 \exp \left( \frac{c(\Omega)}{\bar{\nu}} A_6^2 \right) + \exp \left( -\frac{\bar{\nu} T}{2} \right) \|\mathbf{u}(kT), \mathbf{K}(kT)\|_{L_2(\Omega)}^2.$$

Iterating the above inequality, we get

$$\|\mathbf{u}((k+1)T), \mathbf{K}((k+1)T)\|_{L_2(\Omega)}^2 \leq \frac{\frac{c(\Omega)}{\bar{\nu}} B_1^2 \exp \left( \frac{c(\Omega)}{\bar{\nu}} A_6^2 \right)}{1 - \exp \left( -\frac{\bar{\nu} T}{2} \right)} + \exp \left( -\frac{\bar{\nu} kT}{2} \right) \|\mathbf{u}(0), \mathbf{K}(0)\|_{L_2(\Omega)}^2 \leq \frac{\gamma}{2} + \frac{\gamma}{2} = \gamma,$$

which proves (4.1)<sub>1</sub>.

Next, we integrate (4.2) with respect to time from  $t = kT$  to  $t = (k+1)T$

$$\begin{aligned} \int_{kT}^{(k+1)T} \|\operatorname{rot} \mathbf{u}(\tau), \operatorname{rot} \mathbf{K}(\tau)\|_{L_2(\Omega)}^2 &\leq \frac{c(\Omega)}{\bar{\nu}} \int_{kT}^{(k+1)T} \|\mathbf{u}(\tau), \mathbf{K}(\tau)\|_{L_2(\Omega)}^2 \|\nabla \mathbf{v}_s(\tau), \nabla \mathbf{H}_s(\tau)\|_{L_3(\Omega)}^2 d\tau \\ &+ \frac{c(\Omega)}{\bar{\nu}} \int_{kT}^{(k+1)T} \|\mathbf{g}(\tau)\|_{L_{\frac{6}{5}}(\Omega)}^2 d\tau + \|\mathbf{u}(kT), \mathbf{K}(kT)\|_{L_2(\Omega)}^2. \end{aligned} \quad (4.4)$$

From (4.3) and (4.1) it follows that

$$\sup_{kT \leq t \leq (k+1)T} \|\mathbf{u}(t), \mathbf{K}(t)\|_{L_2(\Omega)}^2 \leq \frac{c(\Omega)}{\bar{\nu}} B_1^2 \exp \left( \frac{c(\Omega)}{\bar{\nu}} A_6^2 \right) + \gamma \exp \left( \frac{c(\Omega)}{\bar{\nu}} A_6^2 \right) \leq 2\gamma \exp \left( \frac{c(\Omega)}{\bar{\nu}} A_6^2 \right),$$

which proves (4.1)<sub>2</sub>. Using the above inequality in (4.4) ends the proof.  $\square$

**Lemma 4.2.** *Let the assumptions of Lemma 4.1 hold. Suppose that  $\mathbf{g} \in L_2(\Omega^{kT})$ . If  $\|\operatorname{rot} \mathbf{u}(0), \operatorname{rot} \mathbf{K}(0)\|_{L_2(\Omega)}^2 \leq \gamma$  and  $\gamma$  is sufficiently small, then*

$$\begin{aligned}
 & \bullet \quad \|\operatorname{rot} \mathbf{u}(kT), \operatorname{rot} \mathbf{K}(kT)\|_{L_2(\Omega)}^2 \leq \gamma, \\
 & \bullet \quad \sup_{kT \leq t \leq (k+1)T} \|\operatorname{rot} \mathbf{u}(t), \operatorname{rot} \mathbf{K}(t)\|_{L_2(\Omega)}^2 \leq \frac{c(\Omega)}{\bar{\nu}} B_1^2 \exp(A_6^2) + \gamma \exp(A_6^2) \equiv \gamma B_4^2, \\
 & \bullet \quad \int_{kT}^{(k+1)T} \|\operatorname{rot} \mathbf{u}(t), \operatorname{rot} \mathbf{K}(t)\|_{L_2(\Omega)}^2 dt \leq \frac{c(\Omega)}{\bar{\nu}^3} \gamma^2 B_4^4 \gamma B_3^2 + \frac{c(\Omega)}{\bar{\nu}} \gamma B_4^2 \exp(A_6^2) + \frac{c(\Omega)}{\bar{\nu}} B_1^2 + \gamma \equiv \gamma B_5^2
 \end{aligned} \tag{4.5}$$

for  $k \in \mathbb{N}_0$ .

*Proof.* Multiplying (1.11)<sub>1,3</sub> by  $\operatorname{rot}^2 \mathbf{u}$  and  $\operatorname{rot}^2 \mathbf{K}$ , respectively, integrating the result over  $\Omega$  and using the boundary conditions (1.11)<sub>5,6</sub>, we get

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \|\operatorname{rot} \mathbf{u}, \operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^2 + \bar{\nu} \|\operatorname{rot}^2 \mathbf{u}, \operatorname{rot}^2 \mathbf{K}\|_{L_2(\Omega)}^2 \leq \int_{\Omega} (-(\mathbf{u} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{v}_s - (\mathbf{v}_s \cdot \nabla) \mathbf{u} \\
 & + (\mathbf{K} \cdot \nabla) \mathbf{K} + (\mathbf{K} \cdot \nabla) \mathbf{H}_s + (\mathbf{H}_s \cdot \nabla) \mathbf{K}) \cdot \operatorname{rot}^2 \mathbf{u} dx + \int_{\Omega} \mathbf{g} \cdot \operatorname{rot}^2 \mathbf{u} dx - \int_{\Omega} ((\mathbf{u} \cdot \nabla) \mathbf{K} - (\mathbf{u} \cdot \nabla) \mathbf{H}_s \\
 & - (\mathbf{v}_s \cdot \nabla) \mathbf{K} + (\mathbf{K} \cdot \nabla) \mathbf{u} + (\mathbf{K} \cdot \nabla) \mathbf{v}_s + (\mathbf{H}_s \cdot \nabla) \mathbf{u}) \cdot \operatorname{rot}^2 \mathbf{K} dx \equiv I_1 + I_2 + I_3.
 \end{aligned} \tag{4.6}$$

By Lemma 2.5, the Hölder and Young inequalities, we have

$$\begin{aligned}
 I_{11} & \leq \|\mathbf{u}\|_{L_{\infty}(\Omega)} \|\nabla \mathbf{u}\|_{L_2(\Omega)} \|\operatorname{rot}^2 \mathbf{u}\|_{L_2(\Omega)} \leq c(\Omega) \|\operatorname{rot}^2 \mathbf{u}\|_{L_2(\Omega)}^{\frac{3}{2}} \|\operatorname{rot} \mathbf{u}\|_{L_2(\Omega)}^{\frac{3}{2}} \\
 & \leq \epsilon_{11} \|\operatorname{rot}^2 \mathbf{u}\|_{L_2(\Omega)}^2 + c \left( \frac{1}{\epsilon_{11}}, \Omega \right) \|\operatorname{rot} \mathbf{u}\|_{L_2(\Omega)}^6, \\
 I_{12} & \leq \|\operatorname{rot}^2 \mathbf{u}\|_{L_2(\Omega)} \|\mathbf{u}\|_{L_6(\Omega)} \|\nabla \mathbf{v}_s\|_{L_3(\Omega)} \leq c(\Omega) \|\operatorname{rot}^2 \mathbf{u}\|_{L_2(\Omega)} \|\mathbf{u}\|_{L_6(\Omega)} \|\mathbf{v}_s\|_{H^2(\Omega)}^{\frac{2}{3}} \|\mathbf{v}_s\|_{L_2(\Omega)}^{\frac{1}{3}} \\
 & \leq \epsilon_{12} \|\operatorname{rot}^2 \mathbf{u}\|_{L_2(\Omega)}^2 + c \left( \frac{1}{\epsilon_{12}}, \Omega \right) \|\operatorname{rot} \mathbf{u}\|_{L_2(\Omega)}^2 \|\mathbf{v}_s\|_{H^2(\Omega)}^2 \\
 I_{13} & \leq \|\operatorname{rot}^2 \mathbf{u}\|_{L_2(\Omega)} \|\nabla \mathbf{u}\|_{L_2(\Omega)} \|\mathbf{v}_s\|_{L_{\infty}(\Omega)} \leq c(\Omega) \|\operatorname{rot}^2 \mathbf{u}\|_{L_2(\Omega)} \|\nabla \mathbf{u}\|_{L_2(\Omega)} \|\mathbf{v}_s\|_{L_2(\Omega)}^{\frac{1}{2}} \|\mathbf{v}_s\|_{H^2(\Omega)}^{\frac{1}{2}} \\
 & \leq \epsilon_{13} \|\operatorname{rot}^2 \mathbf{u}\|_{L_2(\Omega)}^2 + c \left( \frac{1}{\epsilon_{13}}, \Omega \right) \|\operatorname{rot} \mathbf{u}\|_{L_2(\Omega)}^2 \|\mathbf{v}_s\|_{H^2(\Omega)}^2, \\
 I_{14} & \leq \|\mathbf{K}\|_{L_{\infty}(\Omega)} \|\nabla \mathbf{K}\|_{L_2(\Omega)} \|\operatorname{rot}^2 \mathbf{u}\|_{L_2(\Omega)} \leq \epsilon_{14} \|\operatorname{rot}^2 \mathbf{u}\|_{L_2(\Omega)}^2 + c \left( \frac{1}{\epsilon_{14}}, \Omega \right) \|\mathbf{K}\|_{H^2(\Omega)} \|\mathbf{K}\|_{H^1(\Omega)}^3 \\
 & \leq \epsilon_{14} \left( \|\operatorname{rot}^2 \mathbf{u}\|_{L_2(\Omega)}^2 + \|\operatorname{rot}^2 \mathbf{K}\|_{L_2(\Omega)}^2 \right) \\
 & + c \left( \frac{1}{\epsilon_{14}}, \Omega \right) \|\operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^6,
 \end{aligned}$$

and

$$I_{15} \leq \|\operatorname{rot}^2 \mathbf{u}\|_{L_2(\Omega)} \|\mathbf{K}\|_{L_6(\Omega)} \|\nabla \mathbf{H}_s\|_{L_3(\Omega)},$$

where we proceed as in  $I_{12}$ , thus

$$I_{15} \leq \epsilon_{15} \|\operatorname{rot}^2 \mathbf{u}\|_{L_2(\Omega)}^2 + c \left( \frac{1}{\epsilon_{15}}, \Omega \right) \|\operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^2 \|\mathbf{H}_s\|_{H^2(\Omega)}^2.$$

Next

$$I_{16} \leq \|\operatorname{rot}^2 \mathbf{u}\|_{L_2(\Omega)} \|\nabla \mathbf{K}\|_{L_2(\Omega)} \|\mathbf{H}_s\|_{L_\infty(\Omega)},$$

which we estimate analogously to  $I_{13}$ , therefore

$$I_{16} \leq \epsilon_{16} \|\operatorname{rot}^2 \mathbf{u}\|_{L_2(\Omega)}^2 + c \left( \frac{1}{\epsilon_{16}}, \Omega \right) \|\operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^2 \|\mathbf{H}_s\|_{H^2(\Omega)}^2.$$

For  $I_2$ , we have

$$I_2 \leq \epsilon_2 \|\operatorname{rot}^2 \mathbf{u}\|_{L_2(\Omega)}^2 + \frac{1}{4\epsilon_2} \|\mathbf{g}\|_{L_2(\Omega)}^2.$$

Finally, for terms in  $I_3$ , we have

$$\begin{aligned} I_{31} &\leq \|\mathbf{u}\|_{L_\infty(\Omega)} \|\nabla \mathbf{K}\|_{L_2(\Omega)} \|\operatorname{rot}^2 \mathbf{K}\| \\ &\leq \epsilon_{31} \|\operatorname{rot}^2 \mathbf{K}\|_{L_2(\Omega)}^2 + c \left( \frac{1}{\epsilon_{31}}, \Omega \right) \|\operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^2 \|\mathbf{u}\|_{H^1(\Omega)} \|\mathbf{u}\|_{H^2(\Omega)} \\ &\leq \epsilon_{31} \left( \|\operatorname{rot}^2 \mathbf{K}\|_{L_2(\Omega)}^2 + \|\operatorname{rot}^2 \mathbf{u}\|_{L_2(\Omega)}^2 \right) + c \left( \frac{1}{\epsilon_{31}}, \Omega \right) \|\operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^4 \|\operatorname{rot} \mathbf{u}\|_{L_2(\Omega)}^2 \\ &\leq \epsilon_{31} \left( \|\operatorname{rot}^2 \mathbf{K}\|_{L_2(\Omega)}^2 + \|\operatorname{rot}^2 \mathbf{u}\|_{L_2(\Omega)}^2 \right) + c \left( \frac{1}{\epsilon_{31}}, \Omega \right) \left( \|\operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^6 + \|\operatorname{rot} \mathbf{u}\|_{L_2(\Omega)}^6 \right). \end{aligned}$$

For  $I_{32}$ , we repeat the estimate we derived for  $I_{12}$

$$I_{32} \leq \epsilon_{32} \|\operatorname{rot}^2 \mathbf{K}\|_{L_2(\Omega)}^2 + c \left( \frac{1}{\epsilon_{32}}, \Omega \right) \|\operatorname{rot} \mathbf{u}\|_{L_2(\Omega)}^2 \|\mathbf{H}_s\|_{H^2(\Omega)}^2.$$

The term  $I_{33}$  can be estimated in the same way as  $I_{13}$

$$I_{33} \leq \epsilon_{33} \|\operatorname{rot}^2 \mathbf{K}\|_{L_2(\Omega)}^2 + c \left( \frac{1}{\epsilon_{33}}, \Omega \right) \|\operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^2 \|\mathbf{v}_s\|_{H^2(\Omega)}^2.$$

For  $I_{34}$ , we have

$$\begin{aligned} I_{34} &\leq \|\operatorname{rot}^2 \mathbf{K}\|_{L_2(\Omega)} \|\nabla \mathbf{u}\|_{L_2(\Omega)} \|\mathbf{K}\|_{L_\infty(\Omega)} \leq c(\Omega) \|\operatorname{rot}^2 \mathbf{K}\|_{L_2(\Omega)} \|\operatorname{rot} \mathbf{u}\|_{L_2(\Omega)} \|\mathbf{K}\|_{H^1(\Omega)}^{\frac{1}{2}} \|\mathbf{K}\|_{H^2(\Omega)}^{\frac{1}{2}} \\ &\leq \epsilon_{34} \|\operatorname{rot}^2 \mathbf{K}\|_{L_2(\Omega)}^2 + c \left( \frac{1}{\epsilon_{34}}, \Omega \right) \|\operatorname{rot} \mathbf{u}\|_{L_2(\Omega)}^4 \|\operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^2 \\ &\leq \epsilon_{34} \|\operatorname{rot}^2 \mathbf{K}\|_{L_2(\Omega)}^2 + c \left( \frac{1}{\epsilon_{34}}, \Omega \right) \left( \|\operatorname{rot} \mathbf{u}\|_{L_2(\Omega)}^6 + \|\operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^6 \right). \end{aligned}$$

The last two terms in  $I_3$  are very similar to  $I_{12}$  and  $I_{16}$ , respectively, thus

$$I_{35} \leq \epsilon_{35} \|\operatorname{rot}^2 \mathbf{K}\|_{L_2(\Omega)}^2 + c \left( \frac{1}{\epsilon_{35}}, \Omega \right) \|\operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^2 \|\mathbf{v}_s\|_{H^2(\Omega)}^2$$

and

$$I_{36} \leq \epsilon_{36} \|\operatorname{rot}^2 \mathbf{K}\|_{L_2(\Omega)}^2 + c \left( \frac{1}{\epsilon_{36}}, \Omega \right) \|\operatorname{rot} \mathbf{u}\|_{L_2(\Omega)}^2 \|\mathbf{H}_s\|_{H^2(\Omega)}^2.$$

Finally, if all  $\epsilon_i$  are sufficiently small, we obtain

$$\begin{aligned} \frac{d}{dt} \|\operatorname{rot} \mathbf{u}, \operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^2 + \bar{\nu} \|\operatorname{rot}^2 \mathbf{u}, \operatorname{rot}^2 \mathbf{K}\|_{L_2(\Omega)}^2 &\leq \frac{c(\Omega)}{\bar{\nu}^3} \|\operatorname{rot} \mathbf{u}, \operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^6 \\ &\quad + \frac{c(\Omega)}{\bar{\nu}} \|\operatorname{rot} \mathbf{u}, \operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^2 \|\mathbf{v}_s, \mathbf{H}_s\|_{H^2(\Omega)}^2 \\ &\quad + \frac{c(\Omega)}{\bar{\nu}} \|\mathbf{g}\|_{L_2(\Omega)}^2, \end{aligned} \tag{4.7}$$

By Lemma 2.2 the above inequality implies

$$\begin{aligned} \frac{d}{dt} \|\operatorname{rot} \mathbf{u}, \operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^2 + \bar{\nu} c(\Omega) \|\operatorname{rot} \mathbf{u}, \operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^2 &\leq \frac{c(\Omega)}{\bar{\nu}^3} \|\operatorname{rot} \mathbf{u}, \operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^6 \\ &\quad + \frac{c(\Omega)}{\bar{\nu}} \|\operatorname{rot} \mathbf{u}, \operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^2 \|\mathbf{v}_s, \mathbf{H}_s\|_{H^2(\Omega)}^2 \\ &\quad + \frac{c(\Omega)}{\bar{\nu}} \|\mathbf{g}\|_{L_2(\Omega)}^2, \end{aligned} \quad (4.8)$$

which is equivalent to

$$\begin{aligned} &\frac{d}{dt} \left( \|\operatorname{rot} \mathbf{u}, \operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^2 \exp \left( t \bar{\nu} c(\Omega) - \int_{kT}^t \|\mathbf{v}_s(\tau), \mathbf{H}_s(\tau)\|_{H^2(\Omega)}^2 d\tau \right) \right) \\ &\leq \frac{c(\Omega)}{\bar{\nu}^3} \|\operatorname{rot} \mathbf{u}, \operatorname{rot} \mathbf{K}\|_{L_2(\Omega)}^6 \exp \left( t \bar{\nu} c(\Omega) - \int_{kT}^t \|\mathbf{v}_s(\tau), \mathbf{H}_s(\tau)\|_{H^2(\Omega)}^2 d\tau \right) \\ &\quad + \frac{c(\Omega)}{\bar{\nu}} \|\mathbf{g}\|_{L_2(\Omega)}^2 \exp \left( t \bar{\nu} c(\Omega) - \int_{kT}^t \|\mathbf{v}_s(\tau), \mathbf{H}_s(\tau)\|_{H^2(\Omega)}^2 d\tau \right). \end{aligned}$$

After integrating with respect to time from  $kT$  to  $t \in (kT, (k+1)T)$ , we obtain

$$\begin{aligned} \|\operatorname{rot} \mathbf{u}(t), \operatorname{rot} \mathbf{K}(t)\|_{L_2(\Omega)}^2 &\leq \frac{c(\Omega)}{\bar{\nu}^3} \sup_{kT \leq \tau \leq t} \|\operatorname{rot} \mathbf{u}(\tau), \operatorname{rot} \mathbf{K}(\tau)\|_{L_2(\Omega)}^4 \int_{kT}^t \|\operatorname{rot} \mathbf{u}(\tau), \operatorname{rot} \mathbf{K}(\tau)\|_{L_2(\Omega)}^2 \\ &\quad \cdot \exp \left( \tau \bar{\nu} c(\Omega) - \int_{kT}^{\tau} \|\mathbf{v}_s(t'), \mathbf{H}_s(t')\|_{H^2(\Omega)}^2 dt' \right) d\tau \exp \left( -t \bar{\nu} c(\Omega) + \int_{kT}^t \|\mathbf{v}_s(t), \mathbf{H}_s(t)\|_{H^2(\Omega)}^2 dt \right) \\ &\quad + \frac{c(\Omega)}{\bar{\nu}} \int_{kT}^t \|\mathbf{g}(\tau)\|_{L_2(\Omega)}^2 \exp \left( \tau \bar{\nu} c(\Omega) - \int_{kT}^{\tau} \|\mathbf{v}_s(t'), \mathbf{H}_s(t')\|_{H^2(\Omega)}^2 dt' \right) d\tau \\ &\quad \cdot \exp \left( -t \bar{\nu} c(\Omega) + \int_{kT}^t \|\mathbf{v}_s(t), \mathbf{H}_s(t)\|_{H^2(\Omega)}^2 dt \right) \\ &\quad + \|\operatorname{rot} \mathbf{u}(kT), \operatorname{rot} \mathbf{K}(kT)\|_{L_2(\Omega)}^2 \exp \left( -(t - kT) \bar{\nu} c(\Omega) + \int_{kT}^t \|\mathbf{v}_s(t), \mathbf{H}_s(t)\|_{H^2(\Omega)}^2 dt \right). \end{aligned} \quad (4.9)$$

Using the assumptions, Lemmas 3.2 and 4.1, we get

$$\begin{aligned} \|\operatorname{rot} \mathbf{u}(t), \operatorname{rot} \mathbf{K}(t)\|_{L_2(\Omega)}^2 &\leq \frac{c(\Omega)}{\bar{\nu}^3} \sup_{kT \leq t \leq (k+1)T} \|\operatorname{rot} \mathbf{u}(t), \operatorname{rot} \mathbf{K}(t)\|_{L_2(\Omega)}^4 \gamma B_3^2 \exp(A_6^2) \\ &\quad + \frac{c(\Omega)}{\bar{\nu}} B_1^2 \exp(A_6^2) + \|\operatorname{rot} \mathbf{u}(kT), \operatorname{rot} \mathbf{K}(kT)\|_{L_2(\Omega)}^2 \exp(A_6^2), \end{aligned}$$

which for sufficiently small  $\gamma$  implies

$$\sup_{kT \leq t \leq (k+1)T} \|\operatorname{rot} \mathbf{u}(t), \operatorname{rot} \mathbf{K}(t)\|_{L_2(\Omega)}^2 \leq \frac{c(\Omega)}{\bar{\nu}} B_1^2 \exp(A_6^2) + \|\operatorname{rot} \mathbf{u}(kT), \operatorname{rot} \mathbf{K}(kT)\|_{L_2(\Omega)}^2 \exp(A_6^2). \quad (4.10)$$



Next, we take  $t = (k+1)T$  in (4.9), use the above inequality and the assumptions

$$\begin{aligned} & \|\operatorname{rot} \mathbf{u}((k+1)T), \operatorname{rot} \mathbf{K}((k+1)T)\|_{L_2(\Omega)}^2 \\ & \leq \frac{c(\Omega)}{\bar{\nu}^3} \gamma^2 B_4^4 \gamma B_3^2 \exp(A_6^2) + B_1^2 \exp(A_6^2) + \|\operatorname{rot} \mathbf{u}(kT), \operatorname{rot} \mathbf{K}(kT)\|_{L_2(\Omega)}^2 \exp\left(-\frac{\bar{\nu}T}{2}\right). \end{aligned}$$

Iterating the above inequality, we get

$$\begin{aligned} & \|\operatorname{rot} \mathbf{u}((k+1)T), \operatorname{rot} \mathbf{K}((k+1)T)\|_{L_2(\Omega)}^2 \\ & \leq \frac{\frac{c(\Omega)}{\bar{\nu}^3} \gamma^2 B_4^4 \gamma B_3^2 \exp(A_6^2) + B_1^2 \exp(A_6^2)}{1 - \exp(-\frac{\bar{\nu}T}{2})} + \|\operatorname{rot} \mathbf{u}(0), \operatorname{rot} \mathbf{K}(0)\|_{L_2(\Omega)}^2 \exp\left(-\frac{\bar{\nu}kT}{2}\right) \leq \frac{\gamma}{2} + \frac{\gamma}{2}, \end{aligned}$$

which along with (4.10) proves (4.5)<sub>1,2</sub>.

Finally, we integrate (4.7) with respect to  $t \in [kT, (k+1)T]$ . We obtain

$$\begin{aligned} & \int_{kT}^{(k+1)T} \|\operatorname{rot}^2 \mathbf{u}(t), \operatorname{rot}^2 \mathbf{K}(t)\|_{L_2(\Omega)}^2 dt \leq \frac{c(\Omega)}{\bar{\nu}^3} \int_{kT}^{(k+1)T} \|\operatorname{rot} \mathbf{u}(t), \operatorname{rot} \mathbf{K}(t)\|_{L_2(\Omega)}^6 dt \\ & + \frac{c(\Omega)}{\bar{\nu}} \int_{kT}^{(k+1)T} \|\operatorname{rot} \mathbf{u}(t), \operatorname{rot} \mathbf{K}(t)\|_{L_2(\Omega)}^2 \|\mathbf{v}_s(t), \mathbf{H}_s(t)\|_{H^2(\Omega)}^2 dt + \frac{c(\Omega)}{\bar{\nu}} \int_{kT}^{(k+1)T} \|\mathbf{g}(t)\|_{L_2(\Omega)}^2 dt \\ & + \|\operatorname{rot} \mathbf{u}(kT), \operatorname{rot} \mathbf{K}(kT)\|_{L_2(\Omega)}^2. \end{aligned}$$

From (4.1)<sub>3</sub>, (4.5)<sub>2</sub> and the assumptions, we get

$$\int_{kT}^{(k+1)T} \|\operatorname{rot}^2 \mathbf{u}(t), \operatorname{rot}^2 \mathbf{K}(t)\|_{L_2(\Omega^{kT})}^2 dt \leq \frac{c(\Omega)}{\bar{\nu}^3} \gamma^2 B_4^4 \gamma B_3^2 + \frac{c(\Omega)}{\bar{\nu}} \gamma B_4^2 \exp(A_6^2) + \frac{c(\Omega)}{\bar{\nu}} B_1^2 + \gamma.$$

This ends the proof.  $\square$

**Remark 4.3.** If we assume that  $(\mathbf{u}, \mathbf{K})$  is a solution to (1.11) and the assumptions of Lemma 4.2 hold, then we can easily show

$$\begin{aligned} & \|\mathbf{u}\|_{W_2^{2,1}(\Omega^{kT})} + \|\mathbf{K}\|_{W_2^{2,1}(\Omega^{kT})} + \|\nabla \sigma\|_{L_2(\Omega^{kT})} \\ & \leq \sqrt{\gamma} c(\Omega) (B_4 + B_5) (\sqrt{\gamma} c(\Omega) (B_4 + B_5) B_3 + A_7) + c_2 \sqrt{\gamma} B_3 A_7^2 + B_1 + \sqrt{\gamma} \equiv \sqrt{\gamma} B_6. \end{aligned} \quad (4.11)$$

Indeed, using Lemmas 2.3 and 2.4, we have

$$\begin{aligned} & \|\mathbf{u}\|_{W_2^{2,1}(\Omega^{kT})} + \|\mathbf{K}\|_{W_2^{2,1}(\Omega^{kT})} + \|\nabla \sigma\|_{L_2(\Omega^{kT})} \leq \|(\mathbf{u} \cdot \nabla) \mathbf{u}\|_{L_2(\Omega^{kT})} + \|(\mathbf{u} \cdot \nabla) \mathbf{v}_s\|_{L_2(\Omega^{kT})} \\ & + \|(\mathbf{v}_s \cdot \nabla) \mathbf{u}\|_{L_2(\Omega^{kT})} + \|(\mathbf{K} \cdot \nabla) \mathbf{K}\|_{L_2(\Omega^{kT})} + \|(\mathbf{K} \cdot \nabla) \mathbf{H}_s\|_{L_2(\Omega^{kT})} + \|(\mathbf{H}_s \cdot \nabla) \mathbf{K}\|_{L_2(\Omega^{kT})} + \|\mathbf{g}\|_{L_2(\Omega^{kT})} \\ & + \|(\mathbf{u} \cdot \nabla) \mathbf{K}\|_{L_2(\Omega^{kT})} + \|(\mathbf{u} \cdot \nabla) \mathbf{H}_s\|_{L_2(\Omega^{kT})} + \|(\mathbf{v}_s \cdot \nabla) \mathbf{K}\|_{L_2(\Omega^{kT})} + \|(\mathbf{K} \cdot \nabla) \mathbf{u}\|_{L_2(\Omega^{kT})} \\ & + \|(\mathbf{K} \cdot \nabla) \mathbf{v}_s\|_{L_2(\Omega^{kT})} + \|(\mathbf{H}_s \cdot \nabla) \mathbf{u}\|_{L_2(\Omega^{kT})} + \|\mathbf{u}(kT)\|_{H^1(\Omega)} + \|\mathbf{K}(kT)\|_{H^1(\Omega)}. \end{aligned} \quad (4.12)$$

By the Hölder inequality, we get

$$\begin{aligned} & \|\mathbf{u}\|_{W_2^{2,1}(\Omega^{kT})} + \|\mathbf{K}\|_{W_2^{2,1}(\Omega^{kT})} + \|\nabla \sigma\|_{L_2(\Omega^{kT})} \leq \left( \|\mathbf{u}\|_{L_{10}(\Omega^{kT})} + \|\mathbf{K}\|_{L_{10}(\Omega^{kT})} \right) \\ & \cdot \left( \|\nabla \mathbf{u}\|_{L_{\frac{5}{2}}(\Omega^{kT})} + \|\nabla \mathbf{K}\|_{L_{\frac{5}{2}}(\Omega^{kT})} + \|\nabla \mathbf{v}_s\|_{L_{\frac{5}{2}}(\Omega^{kT})} + \|\nabla \mathbf{H}_s\|_{L_{\frac{5}{2}}(\Omega^{kT})} \right) + \left( \|\mathbf{v}_s\|_{L_{10}(\Omega^{kT})} + \|\mathbf{H}_s\|_{L_{10}(\Omega^{kT})} \right) \\ & \cdot \left( \|\nabla \mathbf{u}\|_{L_{\frac{5}{2}}(\Omega^{kT})} + \|\nabla \mathbf{K}\|_{L_{\frac{5}{2}}(\Omega^{kT})} \right) + \|\mathbf{g}\|_{L_2(\Omega^{kT})} + \|\mathbf{u}(kT)\|_{H^1(\Omega)} + \|\mathbf{K}(kT)\|_{H^1(\Omega)}. \end{aligned} \quad (4.13)$$

From (2.5), Lemma 4.2 and Remark 3.3, we infer that

$$\begin{aligned} & \left( \|\mathbf{u}\|_{L_{10}(\Omega^{kT})} + \|\mathbf{K}\|_{L_{10}(\Omega^{kT})} \right) \left( \|\nabla \mathbf{u}\|_{L_{\frac{5}{2}}(\Omega^{kT})} + \|\nabla \mathbf{K}\|_{L_{\frac{5}{2}}(\Omega^{kT})} + \|\nabla \mathbf{v}_s\|_{L_{\frac{5}{2}}(\Omega^{kT})} + \|\nabla \mathbf{H}_s\|_{L_{\frac{5}{2}}(\Omega^{kT})} \right) \\ & \leq \sqrt{\gamma} c(\Omega) (B_4 + B_5) \left( \|\nabla \mathbf{u}\|_{L_{\frac{5}{2}}(\Omega^{kT})} + \|\nabla \mathbf{K}\|_{L_{\frac{5}{2}}(\Omega^{kT})} + A_7 \right). \end{aligned}$$

Similarly, by (2.6), Lemmas 4.1 and 2.1 we have

$$\begin{aligned} \|\nabla \mathbf{u}\|_{L_{\frac{5}{2}}(\Omega^{kT})} + \|\nabla \mathbf{K}\|_{L_{\frac{5}{2}}(\Omega^{kT})} & \leq c_1 \epsilon \left( \|\mathbf{u}\|_{W_2^{2,1}(\Omega^{kT})} + \|\mathbf{K}\|_{W_2^{2,1}(\Omega^{kT})} \right) + c_2 \epsilon^{-1} \left( \|\mathbf{u}\|_{L_2(\Omega^{kT})} + \|\mathbf{K}\|_{L_2(\Omega^{kT})} \right) \\ & \leq c_1 \epsilon \left( \|\mathbf{u}\|_{W_2^{2,1}(\Omega^{kT})} + \|\mathbf{K}\|_{W_2^{2,1}(\Omega^{kT})} \right) + c_2 \epsilon^{-1} \sqrt{\gamma} B_3. \end{aligned}$$

Using the above inequalities in (4.13) yields

$$\begin{aligned} & \|\mathbf{u}\|_{W_2^{2,1}(\Omega^{kT})} + \|\mathbf{K}\|_{W_2^{2,1}(\Omega^{kT})} + \|\nabla \sigma\|_{L_2(\Omega^{kT})} \\ & \leq \sqrt{\gamma} c(\Omega) (B_4 + B_5) (\sqrt{\gamma} c(\Omega) (B_4 + B_5) B_3 + A_7) + c_2 \sqrt{\gamma} B_3 A_7^2 + B_1 + \sqrt{\gamma}. \end{aligned}$$

*Proof of Theorem 2.* The proof of the existence of solutions to (1.11) is based on the Leray–Schauder fixed point theorem. We follow the idea from [24, Sect. 10].

First, we rewrite (1.11) in the following form

$$\begin{aligned} \mathbf{u}_t - \nu \Delta \mathbf{u} + \nabla \sigma &= \lambda \left( -(\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} - (\bar{\mathbf{u}} \cdot \nabla) \mathbf{v}_s - (\mathbf{v}_s \cdot \nabla) \bar{\mathbf{u}} \right. && \text{in } \Omega^{kT}, \\ &\quad \left. + (\bar{\mathbf{K}} \cdot \nabla) \bar{\mathbf{K}} + (\bar{\mathbf{K}} \cdot \nabla) \mathbf{H}_s + (\mathbf{H}_s \cdot \nabla) \bar{\mathbf{K}} \right) + \mathbf{g} \equiv \mathbf{w}_1 + \mathbf{g} \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega^{kT}, \\ \mathbf{K}_t - \mu \Delta \mathbf{K} &= \lambda \left( -(\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{K}} - (\bar{\mathbf{u}} \cdot \nabla) \mathbf{H}_s - (\mathbf{v}_s \cdot \nabla) \bar{\mathbf{K}} \right. && \text{in } \Omega^{kT}, \\ &\quad \left. + (\bar{\mathbf{K}} \cdot \nabla) \bar{\mathbf{u}} + (\bar{\mathbf{K}} \cdot \nabla) \mathbf{v}_s + (\mathbf{H}_s \cdot \nabla) \bar{\mathbf{u}} \right) \equiv \mathbf{w}_2 \\ \operatorname{div} \mathbf{K} &= 0 && \text{in } \Omega^{kT}, \\ \mathbf{n} \times \operatorname{rot} \mathbf{u} &= \mathbf{0}, \quad \mathbf{n} \cdot \mathbf{u} = 0 && \text{on } S^{kT}, \\ \mathbf{n} \times \operatorname{rot} \mathbf{K} &= \mathbf{0}, \quad \mathbf{n} \cdot \mathbf{K} = 0 && \text{on } S^{kT}, \\ \mathbf{u}|_{t=kT} &= \mathbf{u}(kT), \quad \mathbf{K}|_{t=kT} = \mathbf{K}(kT) && \text{in } \Omega. \end{aligned} \tag{4.14}$$

This way, we introduce a mapping  $\Phi: \mathfrak{M} \times \mathfrak{M} \times [0, 1] \rightarrow \mathfrak{M} \times \mathfrak{M}$ ,  $\Phi(\bar{\mathbf{u}}, \bar{\mathbf{K}}, \lambda) = (\mathbf{u}, \mathbf{K})$ , where we define

$$\mathfrak{M} = \left\{ \mathbf{z}: \Omega^{kT} \rightarrow \mathbb{R}^3 : \|\mathbf{z}\|_{L_{\frac{20}{3}}(\Omega^{kT})} < \infty, \|\nabla \mathbf{z}\|_{L_{\frac{20}{7}}(\Omega^{kT})} < \infty \right\}.$$

If we prove that  $\Phi$  has the following properties:

- (1) for  $\lambda = 0$  there exists a unique solution
- (2)  $\Phi(\cdot, \cdot, \lambda)$ ,  $\lambda > 0$ , is compact and continuous
- (3)  $\Phi(\bar{\mathbf{u}}, \bar{\mathbf{K}}, \cdot)$  is uniformly continuous,
- (4) there exists a bounded subset  $A \times A \subset \mathfrak{M} \times \mathfrak{M}$  such that any fixed point of  $\Phi(\cdot, \cdot, \lambda)$  for some  $\lambda \in [0, 1]$  belongs to  $A \times A$

then  $\Phi(\cdot, \cdot, 1)$  will have at least one fixed point.

**Ad.** (1) This property follows immediately from Lemmas 2.3 and 2.4.

**Ad.** (2) The embedding  $W_2^{2,1}(\Omega^{kT}) \hookrightarrow \mathfrak{M}$  is compact and by Lemmas 2.3 and 2.4, we have

$$\begin{aligned} \|\mathbf{u}, \mathbf{K}\|_{\mathfrak{M}} & \leq c(\Omega) \|\mathbf{u}, \mathbf{K}\|_{W_2^{2,1}(\Omega^{kT})} \\ & \leq c(\Omega) \left( \|\mathbf{w}_1\|_{L_2(\Omega^{kT})} + \|\mathbf{w}_2\|_{L_2(\Omega^{kT})} + \|\mathbf{g}\|_{L_2(\Omega^{kT})} \right) + \|\mathbf{u}(kT)\|_{H^1(\Omega)} + \|\mathbf{K}(kT)\|_{H^1(\Omega)}. \end{aligned}$$

Using the Hölder inequality  $\|(\mathbf{a} \cdot \nabla) \mathbf{b}\|_{L_2(\Omega^{kT})} \leq \|\mathbf{a}\|_{L_{\frac{20}{3}}(\Omega^{kT})} \|\nabla \mathbf{b}\|_{L_{\frac{20}{7}}(\Omega^{kT})}$ , we obtain

$$\begin{aligned} \|\mathbf{w}_1\|_{L_2(\Omega^{kT})} + \|\mathbf{w}_2\|_{L_2(\Omega^{kT})} &\leq \left( \|\bar{\mathbf{u}}\|_{L_{\frac{20}{3}}(\Omega^{kT})} + \|\bar{\mathbf{K}}\|_{L_{\frac{20}{3}}(\Omega^{kT})} + \|\nabla \bar{\mathbf{u}}\|_{L_{\frac{20}{7}}(\Omega^{kT})} + \|\nabla \bar{\mathbf{K}}\|_{L_{\frac{20}{7}}(\Omega^{kT})} \right) \\ &\cdot \left( \|\bar{\mathbf{v}}_s\|_{L_{\frac{20}{3}}(\Omega^{kT})} + \|\bar{\mathbf{H}}_s\|_{L_{\frac{20}{3}}(\Omega^{kT})} + \|\nabla \bar{\mathbf{v}}_s\|_{L_{\frac{20}{7}}(\Omega^{kT})} + \|\nabla \bar{\mathbf{H}}_s\|_{L_{\frac{20}{7}}(\Omega^{kT})} \right) \leq c(\Omega) \|\bar{\mathbf{u}}, \bar{\mathbf{K}}\|_{\mathfrak{M}}, \end{aligned}$$

where the last inequality follows from Lemma 3.2 and the embedding theorem. Thus,

$$\|\mathbf{u}, \mathbf{K}\|_{\mathfrak{M}} \leq c(\Omega) \|\bar{\mathbf{u}}, \bar{\mathbf{K}}\|_{\mathfrak{M}} + \|\mathbf{g}\|_{L_2(\Omega^{kT})} + \|\mathbf{u}(kT)\|_{H^1(\Omega)} + \|\mathbf{K}(kT)\|_{H^1(\Omega)}.$$

This justifies the compactness of  $\Phi$ .

To prove the continuity of  $\Phi$ , we take two different sets of arguments of  $\Phi$ , i.e.  $\Phi(\bar{\mathbf{u}}^1, \bar{\mathbf{K}}^1, \lambda) = (\mathbf{u}^1, \mathbf{K}^1)$  and  $\Phi(\bar{\mathbf{u}}^2, \bar{\mathbf{K}}^2, \lambda) = (\mathbf{u}^2, \mathbf{K}^2)$  and consider the differences  $\mathbf{U} = \mathbf{u}^1 - \mathbf{u}^2$ ,  $\mathbf{N} = \mathbf{K}^1 - \mathbf{K}^2$ ,  $S = \sigma^1 - \sigma^2$ . Then, the triple  $(\mathbf{U}, \mathbf{N}, S)$  satisfies

$$\begin{aligned} \mathbf{U}_t - \nu \Delta \mathbf{U} + \nabla S &= \lambda \left( -(\bar{\mathbf{U}} \cdot \nabla) \bar{\mathbf{u}}^2 - (\bar{\mathbf{u}}^1 \cdot \nabla) \bar{\mathbf{U}} - (\bar{\mathbf{U}} \cdot \nabla) \mathbf{v}_s - (\mathbf{v}_s \cdot \nabla) \bar{\mathbf{U}} \right. \\ &\quad \left. + (\bar{\mathbf{N}} \cdot \nabla) \bar{\mathbf{K}}^2 + (\bar{\mathbf{K}}^1 \cdot \nabla) \bar{\mathbf{N}} + (\bar{\mathbf{N}} \cdot \nabla) \mathbf{H}_s + (\mathbf{H}_s \cdot \nabla) \bar{\mathbf{N}} \right) \quad \text{in } \Omega^{kT}, \\ \operatorname{div} \mathbf{U} &= 0 \quad \text{in } \Omega^{kT}, \\ \mathbf{N}_t - \mu \Delta \mathbf{N} &= \lambda \left( -(\bar{\mathbf{U}} \cdot \nabla) \bar{\mathbf{K}}^2 - (\bar{\mathbf{u}}^1 \cdot \nabla) \bar{\mathbf{N}} - (\bar{\mathbf{U}} \cdot \nabla) \mathbf{H}_s - (\mathbf{v}_s \cdot \nabla) \bar{\mathbf{N}} \right. \\ &\quad \left. + (\bar{\mathbf{N}} \cdot \nabla) \bar{\mathbf{u}}^2 + (\bar{\mathbf{K}}^1 \cdot \nabla) \bar{\mathbf{U}} + (\bar{\mathbf{N}} \cdot \nabla) \mathbf{v}_s + (\mathbf{H}_s \cdot \nabla) \bar{\mathbf{U}} \right) \quad \text{in } \Omega^{kT}, \\ \operatorname{div} \mathbf{N} &= 0 \quad \text{in } \Omega^{kT}, \\ \mathbf{n} \times \operatorname{rot} \mathbf{U} &= \mathbf{0}, \quad \mathbf{n} \cdot \mathbf{U} = 0 \quad \text{on } S^{kT}, \\ \mathbf{n} \times \operatorname{rot} \mathbf{N} &= \mathbf{0}, \quad \mathbf{n} \cdot \mathbf{N} = 0 \quad \text{on } S^{kT}, \\ \mathbf{U}|_{t=kT} &= \mathbf{0}, \quad \mathbf{N}|_{t=kT} = \mathbf{0} \quad \text{in } \Omega. \end{aligned} \tag{4.15}$$

By Lemmas 2.3, 2.4 and the embedding  $W_2^{2,1}(\Omega^{kT}) \hookrightarrow \mathfrak{M}$  and Remarks 3.3 and 4.3, we have

$$\begin{aligned} \|\mathbf{U}, \mathbf{N}\|_{\mathfrak{M}} &\leq c(\Omega) \left( \|\bar{\mathbf{U}}\|_{L_{\frac{20}{3}}(\Omega^{kT})} + \|\bar{\mathbf{N}}\|_{L_{\frac{20}{3}}(\Omega^{kT})} + \|\nabla \bar{\mathbf{U}}\|_{L_{\frac{20}{7}}(\Omega^{kT})} + \|\nabla \bar{\mathbf{N}}\|_{L_{\frac{20}{7}}(\Omega^{kT})} \right) \\ &\cdot \left( \|\bar{\mathbf{u}}^1\|_{L_{\frac{20}{3}}(\Omega^{kT})} + \|\mathbf{v}_s\|_{L_{\frac{20}{3}}(\Omega^{kT})} + \|\mathbf{H}_s\|_{L_{\frac{20}{3}}(\Omega^{kT})} + \|\bar{\mathbf{K}}^1\|_{L_{\frac{20}{3}}(\Omega^{kT})} \right. \\ &\quad \left. + \|\nabla \bar{\mathbf{u}}^2\|_{L_{\frac{20}{7}}(\Omega^{kT})} + \|\nabla \mathbf{v}_s\|_{L_{\frac{20}{7}}(\Omega^{kT})} + \|\nabla \mathbf{H}_s\|_{L_{\frac{20}{7}}(\Omega^{kT})} + \|\nabla \bar{\mathbf{K}}^2\|_{L_{\frac{20}{7}}(\Omega^{kT})} \right) \\ &\leq c(\Omega) (\sqrt{\gamma} B_6 + A_7) \|\bar{\mathbf{U}}, \bar{\mathbf{N}}\|_{\mathfrak{M}}, \end{aligned}$$

which justifies the continuity of  $\Phi$ .

**Ad. (3)** This property is evident.

**Ad. (4)** We verified this condition in Remark 4.3.

So far, we have the existence of at least one solution to (1.11). To prove its uniqueness let us assume that there exists another solution. If we introduce the differences between these solutions  $(\mathbf{U}, \mathbf{N}, S) = (\mathbf{u}^1, \mathbf{K}^1, \sigma^1) - (\mathbf{u}^2, \mathbf{K}^2, \sigma^2)$ , then the triple  $(\mathbf{U}, \mathbf{N}, S)$  will satisfy a system of equations which is analogous to (4.15). From energy estimates for that system, we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\mathbf{U}, \mathbf{N}\|_{L_2(\Omega)}^2 + \nu \|\operatorname{rot} \mathbf{U}\|_{L_2(\Omega)}^2 + \mu \|\operatorname{rot} \mathbf{N}\|_{L_2(\Omega)}^2 = - \int_{\Omega} (\mathbf{U} \cdot \nabla) \mathbf{u}^2 \cdot \mathbf{U} d\mathbf{x} \\
& - \int_{\Omega} (\mathbf{U} \cdot \nabla) \mathbf{v}_s \cdot \mathbf{U} d\mathbf{x} + \int_{\Omega} (\mathbf{N} \cdot \nabla) \mathbf{K}^2 \cdot \mathbf{U} d\mathbf{x} + \int_{\Omega} (\mathbf{K}^1 \cdot \nabla) \mathbf{N} \cdot \mathbf{U} d\mathbf{x} + \int_{\Omega} (\mathbf{N} \cdot \nabla) \mathbf{H}_s \cdot \mathbf{U} d\mathbf{x} \\
& + \int_{\Omega} (\mathbf{H}_s \cdot \nabla) \mathbf{N} \cdot \mathbf{U} d\mathbf{x} - \int_{\Omega} (\mathbf{U} \cdot \nabla) \mathbf{K}^2 \cdot \mathbf{N} d\mathbf{x} - \int_{\Omega} (\mathbf{U} \cdot \nabla) \mathbf{H}_s \cdot \mathbf{N} d\mathbf{x} + \int_{\Omega} (\mathbf{N} \cdot \nabla) \mathbf{u}^2 \cdot \mathbf{N} d\mathbf{x} \\
& + \int_{\Omega} (\mathbf{K}^1 \cdot \nabla) \mathbf{U} \cdot \mathbf{N} d\mathbf{x} + \int_{\Omega} (\mathbf{N} \cdot \nabla) \mathbf{v}_s \cdot \mathbf{N} d\mathbf{x} + \int_{\Omega} (\mathbf{H}_s \cdot \nabla) \mathbf{U} \cdot \mathbf{N} d\mathbf{x}.
\end{aligned}$$

By the Hölder and Young inequalities, we obtain

$$\begin{aligned}
& \frac{d}{dt} \|\mathbf{U}, \mathbf{N}\|_{L_2(\Omega)}^2 + \nu \|\operatorname{rot} \mathbf{U}\|_{L_2(\Omega)}^2 + \mu \|\operatorname{rot} \mathbf{N}\|_{L_2(\Omega)}^2 \leq c(\Omega) \|\mathbf{U}, \mathbf{N}\|_{L_2(\Omega)}^2 \\
& \cdot \left( \|\nabla \mathbf{u}^2\|_{L_3(\Omega)}^2 + \|\nabla \mathbf{v}_s\|_{L_3(\Omega)}^2 + \|\nabla \mathbf{K}^2\|_{L_3(\Omega)}^2 + \|\mathbf{K}^1\|_{L_\infty(\Omega)}^2 + \|\nabla \mathbf{H}_s\|_{L_3(\Omega)}^2 + \|\mathbf{H}_s\|_{L_\infty(\Omega)}^2 \right).
\end{aligned}$$

Utilizing the Gronwall inequality, we get

$$\begin{aligned}
\|\mathbf{U}(t), \mathbf{N}(t)\|_{L_2(\Omega)}^2 & \leq \exp \left( \int_{kT}^t \left( \|\nabla \mathbf{u}^2\|_{L_3(\Omega)}^2 + \|\nabla \mathbf{v}_s\|_{L_3(\Omega)}^2 + \|\nabla \mathbf{K}^2\|_{L_3(\Omega)}^2 + \|\mathbf{K}^1\|_{L_\infty(\Omega)}^2 \right. \right. \\
& \left. \left. + \|\nabla \mathbf{H}_s\|_{L_3(\Omega)}^2 + \|\mathbf{H}_s\|_{L_\infty(\Omega)}^2 \right) d\tau \right) \|\mathbf{U}(kT), \mathbf{N}(kT)\|_{L_2(\Omega)}^2 = 0,
\end{aligned}$$

which implies  $\mathbf{U}(t) = \mathbf{0}$  and  $\mathbf{N}(t) = \mathbf{0}$  a.e. This concludes the proof.  $\square$

## 5. Proof of Theorem 3

The proof follows immediately from Lemmas 3.2, 4.2 and Theorem 2.

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